Probability and Stochastic Processes
Lecture 1: Set Theory

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What are Sets?

Definition

A well-defined collection of objects is called a set. The constituent objects of a set are called its elements.

Examples

(a) The set of all natural numbers, denoted as $\mathbb{N} = \{1, 2, \cdots\}$.
(b) The set of all integers, denoted as $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$.
(c) The set of all real numbers, denoted as $\mathbb{R}$.
(d) The set of all rationals, denoted as $\mathbb{Q}$.
(e) The collection of all continents $\{\text{Asia}, \cdots, \text{Antarctica}\}$.

Basic Definitions

Notations

(1) If a is an element of A, we write $a \in A$. If a is not an element of A, we write $a \notin A$.

(2) A set can be defined in two ways.

  (a) Listing all elements, e.g. $A = \{a, b, c\}$.

  (b) Using a common property of its elements, e.g., $A = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.

(3) Discrete sets are written using curly brackets, e.g., $\{0, 1\}$.

(4) Continuous intervals are written as follows.

  (a) open intervals: $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

  (b) closed intervals: $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.

  (c) open-closed intervals: $(0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$.

Basic Definitions

Subsets

If B contains all elements of A, then A is called a subset of B, (denoted as $A \subset B$), and B is called a superset of A. Formally, $A \subset B$ if $a \in A$ implies $a \in B$.

(a) If $A = \{a, b, c\}$ and $B = \{a, b, c, d, e\}$, then $A \subset B$.

(b) If $A = \{2, 3, 4\}$ and $B = \{3, 4, 5\}$, then neither $A \subset B$ nor $B \subset A$.

- If A has at least one element that is not a part of B, then $A \not\subset B$.

Equality of Sets

Two sets A and B are called equal, denoted as $A = B$ if $A \subset B$ and $B \subset A$. If A and B are not equal sets, then we write $A \neq B$.

Basic Definitions

Proper Subset

A set A is called a proper subset of another set B if $A \subset B$ and $B \not\subset A$.

Universal Set

In a given context, a universal set, U, contains all elements under consideration.

(a) While dealing with sets of integers, $\mathbb{Z}$ can be considered the universal set.

(b) While dealing with open intervals, $\mathbb{R}$ can be considered the universal set.

Empty Set

A set, ϕ, is called empty if it does not contain any element.

Basic Definitions

Proposition 1.1

An empty set, ϕ, is a subset of any set.

Proof of Proposition 1.1

If ϕ is not a subset of another set A, then ϕ must have at least one element that is not contained in A. This is impossible since ϕ is empty.

Power Set

The power set of a set A, denoted as 2^A, is the collection of all subsets of A.

- If $A = \{a, b\}$, then $2^A = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.

Set Operations

Complement of a Set

The complement of A, denoted as A^c, contains all elements of the universal set U that are not elements of A. Formally, $A^c = \{x \in U \mid x \notin A\}$. For example:

(a) If $U = [0, 1]$ and $A = (0.5, 1]$, then $A^c = [0, 0.5]$.

(b) If $U = \{a, b, c\}$ and $A = \{a, c\}$, then $A^c = \{b\}$.

Proposition 1.2

For any set A, the following holds: $(A^c)^c = A$.

Proof of Proposition 1.2

Note that $x \in A \iff x \notin A^c \iff x \in (A^c)^c$. Thus, $A \subset (A^c)^c$ and $(A^c)^c \subset A$.

Set Operations

Union of Sets

The union of A and B, denoted as $A \cup B$, contains elements that are either in A or in B. Formally, $A \cup B = \{x|\ x \in A \text{ or } x \in B\}$. More generally, if I is an arbitrary index set, then

$$\bigcup_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in I\}$$

Examples

(a) If $A = \{a, b\}$ and $B = \{b, c, d\}$, then $A \cup B = \{a, b, c, d\}$.

(b) If $k\mathbb{N}$ denotes the set of natural numbers divisible by k, then $2\mathbb{N} \cup 4\mathbb{N} = 2\mathbb{N}$.

(c) If $A_n = [0, \frac{1}{n})$, $\forall n \in \mathbb{N}$, then $\cup_{n \in \mathbb{N}} A_n = [0, 1)$.

Set Operations

Intersection of Sets

The intersection of A and B, denoted as $A \cap B$, contains elements common to both A and B, i.e., $A \cap B = \{x | \, x \in A \text{ and } x \in B\}$. More generally, if I is an arbitrary index set, then

$$\bigcap_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for all } \alpha \in I\}$$

Examples

(a) If $A = \{a, b\}$ and $B = \{b, c, d\}$, then $A \cap B = \{b\}$.

(b) If $k\mathbb{N}$ denotes the set of natural numbers divisible by k, then $2\mathbb{N} \cap 3\mathbb{N} = 6\mathbb{N}$.

(c) If $A_n = [0, \frac{1}{n}), \forall n \in \mathbb{N}$, then $\cap_{n \in \mathbb{N}} A_n = \{0\}$.

Set Operations

Set Difference

For two arbitrary sets A and B, the set difference $A \setminus B$ is defined as the collection of all elements of A that are not included in B. Formally, $A \setminus B = A \cap B^c$.

- If $A = \{a, b, c\}$, and $B = \{b, d\}$, then $A \setminus B = \{a, c\}$.

Cartesian Product

The Cartesian product of two sets A and B is defined as:

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

- If $A = \{a, b\}$ and $B = \{c, d\}$, then $A \times B = \{(a, c), (b, c), (a, d), (b, d)\}$.

Some Properties

Proposition 1.3

If A and $\{B_\alpha\}_{\alpha\in I}$ are arbitrary sets, then the following holds.

(a) $A \cup \left(\cup_{\alpha\in I} B_\alpha\right) = \cup_{\alpha\in I} \left(A \cup B_\alpha\right)$ (Associative Law)

(b) $A \cap \left(\cap_{\alpha\in I} B_\alpha\right) = \cap_{\alpha\in I} \left(A \cap B_\alpha\right)$ (Associative Law)

(c) $A \cap \left(\cup_{\alpha\in I} B_\alpha\right) = \cup_{\alpha\in I} \left(A \cap B_\alpha\right)$ (Distributive Law)

(d) $A \cup \left(\cap_{\alpha\in I} B_\alpha\right) = \cap_{\alpha\in I} \left(A \cup B_\alpha\right)$ (Distributive Law)

Example

If A, B, C are arbitrary, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Some Properties

Proof of Proposition 1.3(c)

Note the following implications.

$$
\begin{aligned}
x \in A \cap \left(\cup_{\alpha \in I} B_\alpha\right) &\iff x \in A \text{ and } x \in \cup_{\alpha \in I} B_\alpha \\
&\iff x \in A \text{ and } \{x \in B_\alpha \text{ for some } \alpha \in I\} \\
&\iff \{x \in A \text{ and } x \in B_\alpha\} \text{ for some } \alpha \in I \\
&\iff x \in A \cap B_\alpha \text{ for some } \alpha \in I \\
&\iff x \in \cup_{\alpha \in I} (A \cap B_\alpha)
\end{aligned}
$$

- Proofs of other propositions are left as an exercise.

De Morgan's Laws

Proposition 1.4

For arbitrary sets $\{A_\alpha\}_{\alpha \in I}$, the following holds.

(a) $(\cup_{\alpha \in I} A_\alpha)^c = \cap_{\alpha \in I} A_\alpha^c$

(b) $(\cap_{\alpha \in I} A_\alpha)^c = \cup_{\alpha \in I} A_\alpha^c$

Proof of Proposition 1.4(a)

$$
\begin{aligned}
x \in (\cup_{\alpha \in I} A_\alpha)^c &\iff x \notin \cup_{\alpha \in I} A_\alpha \\
&\iff x \notin A_\alpha \text{ for all } \alpha \in I \\
&\iff x \in A_\alpha^c \text{ for all } \alpha \in I \iff x \in \cap_{\alpha \in I} A_\alpha^c
\end{aligned}
$$

Basic Definitions

Cardinality of a Finite Set

The cardinality of a finite set A, denoted as $|A|$, is the number of elements of A.

- For example, if $A = \{a, b, c\}$, then $|A| = 3$.

Notes

- Although cardinality is only defined for a finite set, there are ways to extend the notion of cardinality to infinite sets, as we will see later.
- It is easy to verify that $|A \times B| = |A||B|$ and $|2^A| = 2^{|A|}$.
- If $A \cap B = \phi$, then $|A \cup B| = |A| + |B|$.

Some Properties

Proposition 1.5

If A and B are arbitrary finite sets, then the following holds.

(a) $(A \cap B) \cap (A \setminus B) = \phi$.

(b) $(A \cap B) \cup (A \setminus B) = A$.

(c) $A \cap (B \setminus A) = \phi$.

(d) $A \cup (B \setminus A) = A \cup B$.

(e) $|A \setminus B| = |A| - |A \cap B|$.

(f) $|A \cup B| = |A| + |B| - |A \cap B|$.

Some Properties

Proof of Proposition 1.5

Let U be the universal set. We have the following.

(a) $(A \cap B) \cap (A \setminus B) = (A \cap B) \cap (A \cap B^c) = A \cap (B \cap B^c) = A \cap \phi = \phi$.

(b) $(A \cap B) \cup (A \setminus B) = (A \cap B) \cup (A \cap B^c) = A \cap (B \cup B^c) = A \cap U = A$.

(c) $A \cap (B \setminus A) = A \cap (B \cap A^c) = B \cap (A \cap A^c) = B \cap \phi = \phi$.

(d) $A \cup (B \setminus A) = A \cup (B \cap A^c) = (A \cup B) \cap (A \cup A^c) = (A \cup B) \cap U = A \cup B$.

(e) Due to (a) and (b), $|A \cap B| + |A \setminus B| = |A|$.

(f) Due to (c) and (d), $|A \cup B| = |A| + |B \setminus A| = |A| + |B| - |A \cap B|$.

Some Properties

Proposition 1.6

If A, B, C are arbitrary finite sets, then the following holds.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Proposition 1.7 (Inclusion-Exclusion Principle)

If $A_1, \cdots, A_n$ are arbitrary finite sets, then the following holds.

$$|\cup_{i=1}^{n} A_i| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

Some Properties

Proof of Proposition 1.6

Applying Proposition 1.5(*f*), we obtain the following.

$$\begin{aligned}
|A \cup B \cup C| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\
&= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |(A \cap B) \cap (A \cap C)| \\
&= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|
\end{aligned}$$

- The general inclusion-exclusion principle can be proven via induction (exercise).