Probability and Stochastic Processes
Lecture 10: Functions of Random Variables

Prof. Washim Uddin Mondal
Department of Electrical Engineering
Indian Institute of Technology Kanpur

Borel-Measurable Functions

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is called Borel-measurable if $f^{-1}(B) \in \mathcal{B}(\mathbb{R}),\ \forall B \in \mathcal{B}(\mathbb{R})$.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. If $B \in \mathcal{B}(\mathbb{R})$, then

$$\mathcal{B}(\mathbb{R}) \ni f^{-1}(B) = \begin{cases} \phi & \text{if } 0, 1 \notin B \\ (-\infty, 0) & \text{if } 0 \in B, 1 \notin B \\ [0, \infty) & \text{if } 0 \notin B, 1 \in B \\ \mathbb{R} & \text{if } 0, 1 \in B \end{cases}$$

Borel-Measurable Functions

Notes

- A Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$ is essentially a $\mathcal{B}(\mathbb{R})$-measurable function. Such a function can be thought of as a random variable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.
- A large class of functions falls under the category of Borel-measurable functions, e.g., monotonic functions, continuous functions, etc.

Continuous Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is called continuous at $x_0 \in \mathbb{R}$ if $\forall \epsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. We say that f is continuous if it is continuous $\forall x_0 \in \mathbb{R}$.

Borel-Measurable Functions

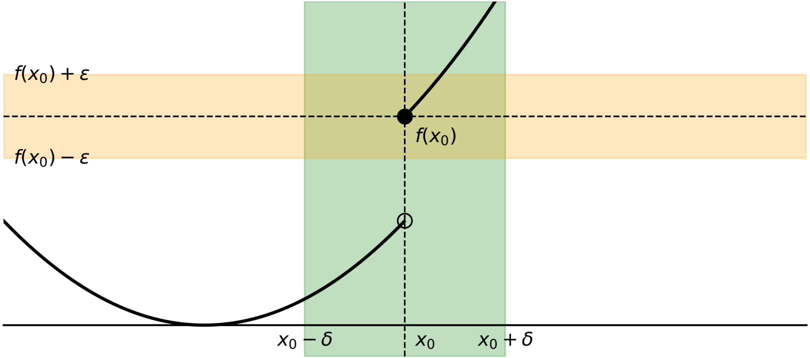

Figure: Pictorial depiction of the $\epsilon - \delta$ definition of continuity.

Borel-Measurable Functions

Example

Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous, where $f(x) = x^2$, $\forall x \in \mathbb{R}$.

Solution

- Fix $x_0 \in \mathbb{R}$ and $\epsilon > 0$. If $x \in \mathbb{R}$ is such that $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| = |x + x_0||x - x_0| < \delta\,(|x - x_0| + 2|x_0|) < \delta(\delta + 2|x_0|)$$

- If $\delta < \min\{1, \epsilon/(1 + 2|x_0|)\}$, then $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Therefore, f is continuous at x_0. Since x_0 is arbitrary, f is continuous everywhere.

Borel-Measurable Functions

Proposition 10.1

If $f : \mathbb{R} \to \mathbb{R}$ has countably many discontinuities, then f is Borel-measurable.

Corollary 10.2

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f is Borel-measurable.

Notes

- A function with uncountably many discontinuities *can* be Borel-measurable (e.g., the Dirichlet function). However, there is no guarantee.
- The proof of Proposition 10.1 is left as an exercise.

Borel-Measurable Functions

Monotonic Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is called

- an increasing function if $f(x_1) \leq f(x_2)$, whenever $x_1 \leq x_2$.
- a decreasing function if $f(x_1) \geq f(x_2)$, whenever $x_1 \leq x_2$.
- a monotonic function if it is either an increasing function or a decreasing function.

Proposition 10.3

Monotonic functions are Borel-measurable.

- Non-monotonic functions *can* be Borel-measurable. However, there is no guarantee.

Functions of Random Variables

Proposition 10.4

If X is a random variable associated with the measurable space $(\Omega, \mathcal{F})$, and $f : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function, then the composite function $f(X) : \Omega \to \mathbb{R}$ defined as $f(X)(\omega) = f(X(\omega)), \forall \omega \in \Omega$ is also a random variable associated with $(\Omega, \mathcal{F})$.

Proof of Proposition 10.4

- Let $B \in \mathcal{B}(\mathbb{R})$. Since f is Borel-measurable, $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$. Therefore,

$$\begin{aligned}
[f(X)]^{-1}(B) &= \{\omega \in \Omega \mid f(X(\omega)) \in B\} \\
&= \{\omega \in \Omega \mid X(\omega) \in f^{-1}(B)\} = X^{-1}(f^{-1}(B)) \in \mathcal{F}
\end{aligned}$$

Functions of Discrete Random Variables

Observation

If X is a discrete random variable and $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable, then $Y = f(X)$ is also a discrete random variable, i.e., if $\mathrm{range}(X)$ is countable, then $\mathrm{range}(Y)$ must be countable as well. Moreover, the PMF of Y is given as follows $\forall y \in \mathbb{R}$.

$$p_Y(y) = \sum_{x_i : f(x_i) = y} p_X(x_i)$$

Example

Let X be a discrete random variable taking values in $\{0, 1, 2, 4\}$ such that $\mathbf{P}(X = 0) = 0.4$, and $\mathbf{P}(X = x) = 0.2$, $x \in \{1, 2, 4\}$. Find the PMF of $Y = \sin(\pi X/4)$.

Functions of Discrete Random Variables

Solution

- The range of Y is $\{0, 1/\sqrt{2}, 1\}$.
- Note that $\{Y = 0\}$ is the same event as $\{X = 0\}$ or $\{X = 4\}$.
- Similarly, $\{Y = 1/\sqrt{2}\}$ is the same event as $\{X = 1\}$.
- Finally, $\{Y = 1\}$ is the same event as $\{X = 2\}$. Therefore,

$$p_Y(y) = \begin{cases} 0.6 & \text{if } y = 0 \\ 0.2 & \text{if } y = \frac{1}{\sqrt{2}} \\ 0.2 & \text{if } y = 1 \end{cases}$$

Functions of Continuous Random Variables

Example

Let $X \sim \mathcal{N}(0,1)$, and $g : \mathbb{R} \to \mathbb{R}$ be such that

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Describe the random variable $Y = g(X)$.

Solution

- $Y = g(X)$ is a discrete random variable with range $\{0, 1\}$ such that $\mathbf{P}(Y = 0) = \mathbf{P}(X < 0) = 0.5$, and $\mathbf{P}(Y = 1) = \mathbf{P}(X \geq 0) = 0.5$.

Functions of Continuous Random Variables

Example

Let $X \sim \exp(1)$, and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 1 & \text{if } 1 \leq x \leq 2 \\ x - 1 & \text{if } 2 < x \end{cases}$$

(a) Find the CDF of $Y = g(X)$.

(b) Is Y a continuous random variable?

(c) Is Y a discrete random variable?

Functions of Continuous Random Variables

Solution

- If $y < 1$, then $\{Y \leq y\}$ is the same event as $\{X \leq y\}$.
- If $y \geq 1$, then $\{Y \leq y\}$ is the same event as $\{X \leq y + 1\}$.

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - \exp(-y) & \text{if } 0 \leq y < 1 \\ 1 - \exp(-(y+1)) & \text{if } 1 \leq y < \infty \end{cases}$$

- Y cannot be a continuous random variable since F_Y is discontinuous at $y = 1$.
- Y cannot be a discrete random variable since the range of Y is not countable.

Functions of Continuous Random Variables

Summary

- Borel-measurable functions of random variables are random variables.
- Functions with countably many discontinuities are Borel-measurable.
- Monotonic functions are Borel-measurable.
- Some random variables are neither discrete nor continuous.
- A (Borel-measurable) function of a discrete random variable must be a discrete random variable.
- A (Borel-measurable) function of a continuous random variable may be a discrete random variable, a continuous random variable, or neither.

Examples

Pareto Distribution

Let $X \sim \exp(\lambda)$, and $Y = ae^X$, where $a > 0$. The random variable Y is said to be Pareto distributed with parameters (a, λ). Find the CDF and the PDF of Y.

Solution

- The CDF can be obtained as follows.

$$F_Y(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ \mathbf{P}(X \leq \ln(y/a)) & \text{if } y > 0 \end{cases} = \begin{cases} 0 & \text{if } y \leq a \\ 1 - (a/y)^\lambda & \text{if } y > a \end{cases}$$

- The PDF is $f_Y(y) = \lambda a^\lambda y^{-(\lambda+1)}$ for $y > a$, and $f_Y(y) = 0$ for $y \leq a$.

Examples

Notes

- Note that if $x_0 > a$, and X is a Pareto random variable with parameters (a, λ), then

$$\mathbf{P}(X \leq x | X > x_0) = \frac{\mathbf{P}(x_0 < X \leq x)}{\mathbf{P}(X > x_0)} = \begin{cases} 1 - (x_0/x)^{\lambda} & \text{if } x > x_0 \\ 0 & \text{otherwise} \end{cases}$$

- Therefore, the conditional distribution of X given $X > x_0$ is a Pareto distribution with parameters (x_0, λ).

Examples

Scale-free nature of Pareto Distribution

Let the minimum income in a country be $a > 0$. Assume that the income of a randomly chosen person is Pareto distributed with parameters (a, λ). Let the income of a certain person A be x_0, where $x_0 > a$. Given that another person B earns more than A, what is the probability that B earns at least αx_0, where $\alpha > 1$?

Solution

- Let $X \sim \text{Pareto}(a, \lambda)$ be the income of person B. Clearly, the conditional distribution of X given $X > x_0$ is Pareto with (x_0, λ) parameters.
- Therefore, $\mathbf{P}(X > \alpha x_0 \mid X > x_0) = (x_0/\alpha x_0)^{\lambda} = \alpha^{-\lambda}$.

Examples

Interpretation of the Result

- Note that the above result is independent of x_0.
- No matter how large x_0 is, the fraction of people that earn more than α times x_0 among all the people that earn more than x_0 is a constant number.
- This is called the scale-free behavior of the Pareto distribution, which is typically observed in income and wealth distribution data.

Log-Normal Distribution

Let $X \sim \mathcal{N}(\mu, \sigma^2)$, and $Y = e^X$. The random variable Y is said to be log-normal distributed with (μ, σ^2) parameters. Find the CDF and the PDF of Y.

Examples

Solution

- The CDF can be obtained as follows.

$$F_Y(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \mathbf{P}(X \le \ln(y)) & \text{if } y > 0 \end{cases} = \begin{cases} 0 & \text{if } y \le 0 \\ \Phi((\ln(y) - \mu)/\sigma) & \text{if } y > 0 \end{cases}$$

- The PDF can be computed as follows.

$$f_Y(y) = F_Y'(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\ln(y)-\mu)^2}{2\sigma^2}\right) & \text{if } y > 0 \end{cases}$$

Examples

Notes

- Y is called log-normal because $\log(Y) \sim \mathcal{N}(\mu, \sigma^2)$.

χ^2 Distribution

Let $X \sim \mathcal{N}(0, 1)$, and $Y = X^2$. The random variable Y is said to be χ^2 distributed (with one degree of freedom). Find the CDF and the PDF of Y.

Examples

Solution

- The CDF can be computed as follows.

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \mathbf{P}(-\sqrt{y} \leq X \leq \sqrt{y}) & \text{if } y \geq 0 \end{cases} = \begin{cases} 0 & \text{if } y < 0 \\ \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) & \text{if } y \geq 0 \end{cases}$$

- The PDF can be obtained as follows.

$$f_Y(y) = F'_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \dfrac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} \exp\left(-\dfrac{y}{2}\right) & \text{if } y \geq 0 \end{cases}$$

Examples

Notes

- Note that $Y \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$. Hence, χ^2 is a special case of the Gamma distribution.
- If $Y \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$, then Y is said to be χ^2 distributed with n degrees of freedom. This is often abbreviated as $Y \sim \chi^2_n$.

Rayleigh Distribution

Let $X \sim \exp(\lambda)$. Define $Y = \sqrt{X}$. Find the CDF and the PDF of Y.

Examples

Solution

- The CDF can be computed as follows.

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \mathbf{P}(\sqrt{X} \leq y) & \text{if } y \geq 0 \end{cases} = \begin{cases} 0 & \text{if } y < 0 \\ 1 - e^{-\lambda y^2} & \text{if } y \geq 0 \end{cases}$$

- The PDF can be computed as follows.

$$f_Y(y) = F'_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 2\lambda y e^{-\lambda y^2} & \text{if } y \geq 0 \end{cases}$$

Affine Transformation

Proposition 10.5

Let X be a continuous random variable with CDF F_X and PDF f_X. The CDF and the PDF of $Y = aX + b$, $a \neq 0$, are given as follows.

$$F_Y(y) = \begin{cases} F_X\left(\dfrac{y-b}{a}\right) & \text{if } a > 0 \\ 1 - F_X\left(\dfrac{y-b}{a}\right) & \text{if } a < 0 \end{cases} \quad \text{and} \quad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right), \ \forall y \in \mathbb{R}$$

Affine Transformation

Proof of Proposition 10.5

- If $a > 0$, then $F_Y(y) = \mathbf{P}(aX + b \leq y) = \mathbf{P}(X \leq (y-b)/a) = F_X((y-b)/a)$.
- Differentiating, we obtain $f_Y(y) = \frac{1}{a} f_X((y-b)/a)$.
- If $a < 0$, then the CDF can be computed as follows.

$$\begin{aligned} F_Y(y) &= \mathbf{P}(aX + b \leq y) \\ &= \mathbf{P}(X \geq (y-b)/a) = \mathbf{P}(X > (y-b)/a) = 1 - F_X((y-b)/a) \end{aligned}$$

- Differentiating, we obtain $f_Y(y) = -\frac{1}{a} f_X((y-b)/a)$.

Affine Transformation

Example

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Show that $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$, where $a \neq 0$.

Solution

$$
\begin{aligned}
f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) &= \frac{1}{|a|\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma^2 a^2}} \exp\left(-\frac{(y - a\mu - b)^2}{2\sigma^2 a^2}\right)
\end{aligned}
$$