Probability and Stochastic Processes
Lecture 12: Multiple Random Variables

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Introduction

Motivation

- Consider a doctor trying to assess the condition of a patient by jointly looking at multiple numeric indicators, such as heart rate, body temperature, blood pressure, etc.
- Here, the underlying measurable space $(\Omega, \mathcal{F})$ is inaccessible because $(\Omega, \mathcal{F})$ describes the outcomes of every biochemical processes happening inside the body.
- The multiple indicators can be jointly described as a function $X : \Omega \to \mathbb{R}^n$.
- Measurable space of the observer is $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Hence, every event in $\mathcal{B}(\mathbb{R}^n)$ must translate to an event in $\mathcal{F}$, i.e., $X^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R}^n)$.
- A generalization of real-valued random variables.

Borel σ-algebra of $\mathbb{R}^n$

Definition

The Borel σ-algebra of $\mathbb{R}^n$, denoted as $\mathcal{B}(\mathbb{R}^n)$, is defined as the σ-algebra generated by the collection of subsets of $\mathbb{R}^n$ of the form $(-\infty, a_1] \times \cdots \times (-\infty, a_n]$, where $a_1, \cdots, a_n \in \mathbb{R}$.

Equivalent Definitions

$\mathcal{B}(\mathbb{R}^n)$ is the σ-algebra generated by the collection of subsets of $\mathbb{R}^n$ of the form $B_1 \times \cdots \times B_n$, where B_i, $i \in \{1, \cdots, n\}$ is either of the following forms.

- $(-\infty, a]$, $(-\infty, a)$, $[a, \infty)$, (a, ∞)
- $[a, b)$, $(a, b]$, (a, b), $[a, b]$

where $a, b \in \mathbb{R}$ and $a < b$.

Random Vectors

Definition

A function $X : \Omega \to \mathbb{R}^n$ is called an $\mathbb{R}^n$-valued random vector associated with a measurable space $(\Omega, \mathcal{F})$ if $\forall B \in \mathcal{B}(\mathbb{R}^n)$, $X^{-1}(B) \in \mathcal{F}$. Such a function is also called an $\mathcal{F}/\mathcal{B}(\mathbb{R}^n)$-measurable function, an $\mathcal{F}$-measurable function, or simply a measurable function.

- $\mathbb{R}$-valued random vectors are random variables.

Examples

Let $\Omega = \{a, b, c\}$, $\mathcal{F}_1 = \{\phi, \Omega, \{a\}, \{b, c\}\}$, and $\mathcal{F}_2 = \{\phi, \Omega, \{a, b\}, \{c\}\}$. If $X : \Omega \to \mathbb{R}^2$ is such that $X(a) = X(b) = (1, 2)$, and $X(c) = (3, 3)$, then

- X is not $\mathcal{F}_1$-measurable because $X^{-1}(\{3, 3\}) = \{c\} \notin \mathcal{F}_1$ but it is $\mathcal{F}_2$-measurable.

Multiple Random Variables

Proposition 12.1

Let $(\Omega, \mathcal{F})$ be a measurable space and $X_i : \Omega \to \mathbb{R}$ be arbitrary, $\forall i \in \{1, \cdots, n\}$. Define $X : \Omega \to \mathbb{R}^n$ such that $X(\omega) = (X_1(\omega), \cdots, X_n(\omega))$, $\forall \omega \in \Omega$. The map X is an $\mathbb{R}^n$-valued random vector on $(\Omega, \mathcal{F})$ iff X_i is a random variable on $(\Omega, \mathcal{F})$, $\forall i \in \{1, \cdots, n\}$.

Notes

- In summary, X is $\mathcal{F}/\mathcal{B}(\mathbb{R}^n)$-measurable iff X_i is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable $\forall i \in \{1, \cdots, n\}$.
- The proof is left as an exercise.

Characterization of Random Vectors

Definition

The induced measure of a random vector $X : \Omega \to \mathbb{R}^n$ associated with a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is a function $\mathbf{P}_X : \mathcal{B}(\mathbb{R}^n) \to [0, 1]$ such that $\mathbf{P}_X(B) = \mathbf{P}(X^{-1}(B)), \forall B \in \mathcal{B}(\mathbb{R}^n)$.

- One can show that $\mathbf{P}_X$ is a probability measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.

Definition

If $X = (X_1, \cdots, X_n) : \Omega \to \mathbb{R}^n$ is a random vector associated with a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, then its joint CDF is a function $F_X : \mathbb{R}^n \to [0, 1]$ such that

$$F_X(x_1, \cdots, x_n) = \mathbf{P}(X_1 \le x_1, \cdots, X_n \le x_n), \ \forall (x_1, \cdots, x_n) \in \mathbb{R}^n$$

Discrete Random Vectors

Definition

An $\mathbb{R}^n$-valued random vector X, associated with a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, is called a discrete random vector if $\mathrm{range}(X)$ is countable.

Notes

If $\{X_i\}_{i=1}^n$ are discrete random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, then

- $X = (X_1, \cdots, X_n)$ is an $\mathbb{R}^n$-valued discrete random vector on $(\Omega, \mathcal{F}, \mathbf{P})$.
- The joint probability mass function (PMF) of X is a function $p_X : \mathbb{R}^n \to [0, 1]$ such that the following holds $\forall (x_1, \cdots, x_n) \in \mathbb{R}^n$.

$$p_X(x_1, \cdots, x_n) = \mathbf{P}(X_1 = x_1, \cdots, X_n = x_n)$$

Discrete Random Vectors

Note

- $\sum_{\mathbf{x}_i} p_X(\mathbf{x}_i) = 1$, where $\sum_{\mathbf{x}_i}$ is the sum over the range of X.

Example

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $\mathbf{P} : \mathcal{F} \to [0,1]$ be such that $\mathbf{P}([a,b]) = \mathbf{P}((a,b]) = b - a$, $0 \leq a < b \leq 1$. Consider two $\mathcal{F}$-measurable functions $X_1, X_2 : \Omega \to \mathbb{R}$ such that

$$X_n(\omega) = \begin{cases} 0 & \text{if } \omega \in [0, \frac{1}{n+1}] \\ 1 & \text{if } \omega \in (\frac{1}{1+n}, 1] \end{cases}$$

where $n \in \{1, 2\}$. Find the joint PMF of (X_1, X_2).

Discrete Random Vectors

Solution

- $\{X_1 = 0, X_2 = 0\}$ is the same event as $[0, \frac{1}{2}] \cap [0, \frac{1}{3}] = [0, \frac{1}{3}]$.
- $\{X_1 = 0, X_2 = 1\}$ is the same event as $[0, \frac{1}{2}] \cap (\frac{1}{3}, 1] = (\frac{1}{3}, \frac{1}{2}]$.
- $\{X_1 = 1, X_2 = 1\}$ is the same event as $(\frac{1}{2}, 1] \cap (\frac{1}{3}, 1] = (\frac{1}{2}, 1]$.
- $\{X_1 = 1, X_2 = 0\}$ is the same event as $(\frac{1}{2}, 1] \cap [0, \frac{1}{3}] = \phi$.

	$X_1 = 0$	$X_1 = 1$
$X_2 = 0$	$\frac{1}{3}$	0
$X_2 = 1$	$\frac{1}{6}$	$\frac{1}{2}$

Continuous Random Vector

Definition

A random vector $X : \Omega \to \mathbb{R}^n$, associated with a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called a continuous random vector if its induced measure $\mathbf{P}_X$ can be expressed as follows

$$\mathbf{P}_X(B) = \int_B f_X(\mathbf{x}) \mathrm{d}\mathbf{x}$$

for some $f_X : \mathbb{R}^n \to [0, \infty)$, and $\forall B \in \mathcal{B}(\mathbb{R}^n)$.

- f_X is called the joint probability density function (PDF) of X.
- Since $\mathbf{P}_X(\mathbb{R}^n) = 1$, we must have

$$\int_{\mathbb{R}^n} f_X(\mathbf{x}) \mathrm{d}\mathbf{x} = 1$$

Continuous Random Vector

Notes

- If $X = (X_1, \cdots, X_n)$, and $\Delta x_1, \cdots, \Delta x_n$ are small, then

$$\mathbf{P}(x_1 \le X_1 \le x_1 + \Delta x_1, \cdots, x_n \le X_n \le x_n + \Delta x_n) \approx f_X(x_1, \cdots, x_n)\Delta x_1 \cdots \Delta x_n$$

- Choosing $B = (-\infty, x_1] \times \cdots \times (-\infty, x_n]$, we get

$$F_X(x_1, \cdots, x_n) = \mathbf{P}_X(B) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_X(\mathbf{x})\mathrm{d}\mathbf{x}$$

- If partial derivatives of F_X exist at $(x_1, \cdots, x_n)$, then

$$f_X(x_1, \cdots, x_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_X(x_1, \cdots, x_n)$$

Continuous Random Vector

Example

The joint density function of (X, Y) is given as follows.

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-(x+2y)} & \text{if } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Find the following probabilities.

(a) $\mathbf{P}(X > 1, Y < 1)$

(b) $\mathbf{P}(X < a)$, where $0 < a < \infty$

(c) $\mathbf{P}(X < Y)$

(d) $\mathbf{P}(X + 2Y < 2)$

Continuous Random Vector

Solution

$$\mathbf{P}(X > 1, Y < 1) = \int_1^\infty \int_0^1 2e^{-(x+2y)}\mathrm{d}y\mathrm{d}x = e^{-1}(1 - e^{-2})$$

$$\mathbf{P}(X < a) = \mathbf{P}(X < a, Y < \infty) = \int_0^a \int_0^\infty 2e^{-(x+2y)}\mathrm{d}y\mathrm{d}x = 1 - e^{-a}$$

$$\mathbf{P}(X < Y) = \int_0^\infty \int_x^\infty 2e^{-(x+2y)}\mathrm{d}y\mathrm{d}x = \int_0^\infty e^{-3x}\mathrm{d}x = \frac{1}{3}$$

$$\mathbf{P}(X + 2Y \leq 2) = \int_0^1 \int_0^{2-2y} 2e^{-(x+2y)}\mathrm{d}x\mathrm{d}y = \int_0^1 2e^{-2y}(1 - e^{-2+2y})\mathrm{d}y = 1 - 3e^{-2}$$

Marginal Distribution

Definition

- Let the joint PDF of (X, Y) be $f_{X,Y}$. The marginal CDF of X is expressed below.

$$F_X(x) = \mathbf{P}(X \leq x) = \mathbf{P}(X \leq x, Y \in \mathbb{R}) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x', y) \mathrm{d}y \mathrm{d}x'$$

- The marginal PDF of X is given as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \mathrm{d}y$$

- If X, Y are discrete random variables, then the marginal PMF of X is $\mathbf{P}(X = x)$ $= \sum_{y_i} \mathbf{P}(X = x, Y = y_i), \forall x \in \mathbb{R}^n$, where the sum is taken over the range of Y.

Marginal Distribution

Example

Let the joint PDF of (X_1, X_2) be as follows:

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left\{\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} - \frac{2\rho x_1 x_2}{\sigma_1\sigma_2}\right\}\right)$$

where $x_1, x_2 \in \mathbb{R}$, $\sigma_1^2, \sigma_2^2 > 0$, and $\rho \in (-1, 1)$. Show that $X_1 \sim \mathcal{N}(0, \sigma_1^2)$.

Solution

$$\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2} - \frac{2\rho x_1 x_2}{\sigma_1\sigma_2} = \frac{(1-\rho^2)x_1^2}{\sigma_1^2} + \left(\frac{x_2}{\sigma_2} - \frac{\rho x_1}{\sigma_1}\right)^2$$

Marginal Distribution

Solution (Continued)

- Therefore, $f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\mathrm{d}x_2 = \frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left(-\frac{x_1^2}{2\sigma_1^2}\right) \times J$ where

$$J = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_2^2(1-\rho^2)}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x_2}{\sigma_2}-\frac{\rho x_1}{\sigma_1}\right)^2\right\}\mathrm{d}x_2$$

$$\overset{(a)}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)\mathrm{d}y = 1$$

- Step (a) results from the substitution that $y = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{x_2}{\sigma_2}-\frac{\rho x_1}{\sigma_1}\right)$.

Marginal Distribution

Notes

- The random vector (X_1, X_2) is called a bivariate normal random vector with $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ parameters, where $\rho \in (-1, 1)$, and $\sigma_1^2, \sigma_2^2 > 0$ if its joint PDF is

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\sqrt{(2\pi)^2 \sigma_1^2 \sigma_2^2 (1-\rho^2)}} \times \exp\left(-\frac{1}{2(1-\rho^2)}\left\{\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right\}\right)$$

- It can be shown that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$.

Independent Random Variables

Definition

The random variables $\{X_i\}_{i=1}^n$, associated with the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ are called independent if the following holds $\forall B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R})$.

$$\mathbf{P}(X_1 \in B_1, \cdots, X_n \in B_n) = \mathbf{P}(X_1 \in B_1) \cdots \mathbf{P}(X_n \in B_n)$$

Equivalent Definition

- $X_1, \cdots, X_n$ are independent random variables if

$$F_{X_1,\cdots,X_n}(x_1, \cdots, x_n) = F_{X_1}(x_1) \cdots F_{X_n}(x_n),\ \forall (x_1, \cdots, x_n) \in \mathbb{R}^n$$

Independent Random Variables

Equivalent Definitions

- If $X = (X_1, \cdots, X_n)$ is a discrete random vector, then $X_1, \cdots, X_n$ are called independent random variables if the following holds $\forall (x_1, \cdots, x_n) \in \mathbb{R}^n$.

$$\mathbf{P}(X_1 = x_1, \cdots, X_n = x_n) = \mathbf{P}(X_1 = x_1) \cdots \mathbf{P}(X_n = x_n)$$

- If $X = (X_1, \cdots, X_n)$ is a continuous random vector, then $X_1, \cdots, X_n$ are called independent random variables if the following equality holds $\forall (x_1, \cdots, x_n) \in \mathbb{R}^n$.

$$f_{X_1, \cdots, X_n}(x_1, \cdots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n)$$

Independent Random Variables

Example

Check if the random variables X, Y are independent whose joint PMF is given below.

	$X=1$	$X=2$	$X=3$
$Y=1$	0.1	0	0.25
$Y=2$	0.3	0.35	0

Solution

- Note that $\mathbf{P}(X=2)=0.35$, $\mathbf{P}(Y=1)=0.35$.
- However, $\mathbf{P}(X=2, Y=1)=0 \neq \mathbf{P}(X=2)\mathbf{P}(Y=1)$.

Independent Random Variables

Example

The joint PDF of X, Y are given as follows. Check if they are independent.

$$f_{X,Y}(x,y) = \begin{cases} 6xy^2 & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Solution

- Let $0 \le x \le 1$, $0 \le y \le 1$. Clearly, $f_X(x) = \int_0^1 6xy^2 \mathrm{d}y = 2x$, and $f_Y(y) = \int_0^1 6xy^2 \mathrm{d}x = 3y^2$. Hence, X, Y are independent.

Independent Random Variables

Example

Let A, B, C be independent uniform random variables in $(0, 1)$. Find the probability that the quadratic equation $Ax^2 + Bx + C = 0$ does not have a real root.

Solution

- The joint PDF of A, B, C is given as follows.

$$f_{A,B,C}(a, b, c) = \begin{cases} 1 & \text{if } (a, b, c) \in (0, 1)^3 \\ 0 & \text{otherwise} \end{cases}$$

- We need to find the probability of the event $\{B < 2\sqrt{AC}\}$.

Independent Random Variables

Solution (Continued)

$$
\begin{aligned}
\mathbf{P}(B < 2\sqrt{AC}) &= \mathbf{P}\left(\frac{1}{4} < AC \leq 1\right) + \mathbf{P}\left(B \leq 2\sqrt{AC}, AC \leq \frac{1}{4}\right) \\
&= \mathbf{P}\left(\frac{1}{4} < A < 1, C > \frac{1}{4A}\right) + \mathbf{P}\left(B \leq 2\sqrt{AC}, 0 < A \leq \frac{1}{4}\right) \\
&+ \mathbf{P}\left(B \leq 2\sqrt{AC}, \frac{1}{4} < A < 1, C \leq \frac{1}{4A}\right)
\end{aligned}
$$

Independent Random Variables

Solution (Continued)

$$\mathbf{P}\left(\frac{1}{4} < A < 1, C > \frac{1}{4A}\right) = \int_{\frac{1}{4}}^{1}\int_{\frac{1}{4a}}^{1}\int_{0}^{1} \mathrm{d}b\mathrm{d}c\mathrm{d}a = \int_{\frac{1}{4}}^{1}\left(1 - \frac{1}{4a}\right)\mathrm{d}a = \frac{3}{4} - \frac{1}{2}\ln(2)$$

$$\mathbf{P}\left(B \leq 2\sqrt{AC}, 0 < A \leq \frac{1}{4}\right) = \int_{0}^{\frac{1}{4}}\int_{0}^{1}\int_{0}^{2\sqrt{ac}} \mathrm{d}b\mathrm{d}c\mathrm{d}a$$

$$= 2\int_{0}^{\frac{1}{4}}\sqrt{a}\mathrm{d}a\int_{0}^{1}\sqrt{c}\mathrm{d}c = 2 \times \frac{2}{3} \times \left(\frac{1}{4}\right)^{3/2} \times \frac{2}{3} = \frac{1}{9}$$

Independent Random Variables

Solution (Continued)

$$\mathbf{P}\left(B \leq 2\sqrt{AC}, \frac{1}{4} < A < 1, C \leq \frac{1}{4A}\right)$$

$$= \int_{\frac{1}{4}}^{1} \int_{0}^{\frac{1}{4a}} \int_{0}^{2\sqrt{ac}} \mathrm{d}b\mathrm{d}c\mathrm{d}a = \int_{\frac{1}{4}}^{1} 2\sqrt{a} \int_{0}^{\frac{1}{4a}} \sqrt{c}\mathrm{d}c\mathrm{d}a$$

$$= \int_{\frac{1}{4}}^{1} \frac{4}{3}\sqrt{a}\left(\frac{1}{4a}\right)^{3/2} \mathrm{d}a = \frac{1}{6}\int_{\frac{1}{4}}^{1} \frac{1}{a}\mathrm{d}a = \frac{1}{3}\ln(2)$$

- The required probability is $\frac{3}{4} + \frac{1}{9} - \frac{1}{6}\ln(2) = 0.745$.

Sum of Discrete Random Variables

Proposition 12.2

- Let X, Y be discrete random variables with range $\{0, 1, \cdots\}$. The PMF of $X + Y$ is:

$$\mathbf{P}(X + Y = n) = \sum_{k=0}^{n} \mathbf{P}(X = k, Y = n - k), \quad \forall n \in \{0, 1, \cdots\}$$

- Moreover, if X, Y are independent, then

$$\mathbf{P}(X + Y = n) = \sum_{k=0}^{n} \mathbf{P}(X = k)\mathbf{P}(Y = n - k), \quad \forall n \in \{0, 1, \cdots\}$$

- The same idea can be extended to obtain the PMF of $X_1 + \cdots + X_n$.

Sum of Discrete Random Variables

Example

Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent random variables. Show that $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

Solution

$$\mathbf{P}(X + Y = n) = \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k} = e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!}$$

for any $n \in \{0, 1, \cdots\}$.

Sum of Discrete Random Variables

Example

Let $X \sim \text{Binomial}(m, p)$ and $Y \sim \text{Binomial}(n, p)$ be independent random variables. Show that $X + Y \sim \text{Binomial}(m + n, p)$.

Solution

$$
\begin{aligned}
\mathbf{P}(X + Y = r) &= \sum_{k=0}^{r} \binom{n}{k} p^k (1-p)^{n-k} \binom{m}{r-k} p^{r-k} (1-p)^{m-r+k} \\
&= \binom{m+n}{r} p^r (1-p)^{m+n-r}
\end{aligned}
$$

for any $r \in \{0, 1, \cdots, m + n\}$, where we use the notation that $\binom{a}{b} = 0$ if $a < b$.

Sum of Discrete Random Variables

Notes

- Out of m men and n women, the number of ways r persons can be chosen is $\binom{m+n}{r}$.
- The number of ways $r-k$ men and k women can be chosen is $\binom{m}{r-k}\binom{n}{k}$. Hence,

$$\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{m+n}{r}$$

- Recall that $\mathrm{Bernoulli}(p)$ is the same as $\mathrm{Binomial}(1,p)$. Hence, if $X_i \sim \mathrm{Bernoulli}(p)$, $i \in \{1, \cdots, n\}$ are independent, then

$$X_1 + \cdots + X_n \sim \mathrm{Binomial}(n,p)$$

Sum of Discrete Random Variables

Example

Let $X \sim \mathrm{NB}(r, p)$ and $Y \sim \mathrm{NB}(s, p)$ be independent random variables. Show that $X + Y \sim \mathrm{NB}(r + s, p)$.

Solution

$$\begin{aligned}\mathbf{P}(X+Y=n) &= \sum_{k=0}^{n} \binom{k-1}{r-1} p^r (1-p)^{k-r} \binom{n-k-1}{s-1} p^s (1-p)^{n-k-s} \\ &= \binom{n-1}{r+s-1} p^{r+s} (1-p)^{n-r-s}\end{aligned}$$

where we use the notation that $\binom{a}{b} = 0$ if $a < b$.

Sum of Discrete Random Variables

Notes

- Place $n-1$ distinct boxes in a row. Our goal is to choose $r+s-1$ boxes from this collection, which can be done in $\binom{n-1}{r+s-1}$ ways.
- Let the rth chosen box be the kth one in the row. The $r-1$ boxes on its left can be chosen in $\binom{k-1}{r-1}$ ways whereas the $s-1$ boxes on its right can be chosen in $\binom{n-k-1}{s-1}$ ways. Therefore,

$$\sum_{k=0}^{n} \binom{k-1}{r-1} \binom{n-k-1}{s-1} = \binom{n-1}{r+s-1}$$

- $\mathrm{Geom}(p)$ is identical to $\mathrm{NB}(1,p)$. If $X_i \sim \mathrm{Geom}(p)$, $i \in \{1, \cdots, n\}$ are independent, then $X_1 + \cdots + X_r \sim \mathrm{NB}(r,p)$.

Sum of Continuous Random Variables

Proposition 12.3

- If the joint PDF of X, Y is $f_{X,Y}$, then the PDF of $X+Y$ is given as:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\mathrm{d}x, \ \forall z \in \mathbb{R}$$

- Moreover, if X, Y are independent with PDFs f_X and f_Y respectively, then

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\mathrm{d}x, \ \forall z \in \mathbb{R}$$

- The same idea can be extended to obtain the PDF of $X_1 + \cdots + X_n$.

Sum of Continuous Random Variables

Proof of Proposition 12.3

- If $f_{X,Y}$ is the joint PDF of (X, Y), then the CDF of $X + Y$ is:

$$F_{X+Y}(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{X,Y}(x, y) \mathrm{d}y \mathrm{d}x$$

- Differentiating with respect to z, we get:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x) \mathrm{d}x$$

Sum of Continuous Random Variables

Example

Let X, Y be independent uniform random variables in $(0, 1)$. Find the PDF of $X + Y$.

Solution

- Note that $f_{X+Y}(z) = 0$ if $z \leq 0$ or $z \geq 2$. Choose $z \in (0, 1]$.

$$f_{X+Y}(z) = \int_0^z \mathrm{d}x = z$$

- Choose $z \in (1, 2)$. We have:

$$f_{X+Y}(z) = \int_{z-1}^1 \mathrm{d}x = 2 - z$$

Sum of Continuous Random Variables

Example

If $X_k \sim \mathcal{N}(\mu_k, \sigma_k^2)$, $k \in \{1, 2\}$ are independent, then $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Solution

$$
\begin{aligned}
f_{X_1+X_2}(z) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\left[\frac{(x-\mu_1)^2}{2\sigma_1^2} + \frac{(z-x-\mu_2)^2}{2\sigma_2^2}\right]\right) \mathrm{d}x \\
&\stackrel{(a)}{=} \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\left[\frac{y^2}{2\sigma_1^2} + \frac{(z-\mu_1-\mu_2-y)^2}{2\sigma_2^2}\right]\right) \mathrm{d}y
\end{aligned}
$$

where (a) follows from the substitution that $y = x - \mu_1$.

Sum of Continuous Random Variables

Solution (Continued)

- Let $a = z - \mu_1 - \mu_2$. Note that

$$\begin{aligned}
\frac{y^2}{\sigma_1^2} + \frac{(a-y)^2}{\sigma_2^2} &= \left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right) y^2 - \frac{2a}{\sigma_2^2} y + \frac{a^2}{\sigma_2^2} \\
&= \left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right) \left\{ y^2 - 2a\left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right) y \right\} + \frac{a^2}{\sigma_2^2} \\
&= \left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right) \left\{ y - \left(\frac{a\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right) \right\}^2 + \frac{a^2}{\sigma_1^2+\sigma_2^2}
\end{aligned}$$

Sum of Continuous Random Variables

Solution (Continued)

$$f_{X_1+X_2}(z) = \exp\left(-\frac{(z-\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)}\right) \times J \text{ where}$$

$$J = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\left(\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\right)\left\{y-\left(\frac{a\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)\right\}^2\right) \mathrm{d}y$$

$$\overset{(a)}{=} \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{\sigma_1^2+\sigma_2^2}} \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}$$

where (a) uses $z = \sqrt{\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}}\left\{y - \frac{a\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right\}$.

Sum of Continuous Random Variables

Example

If $X \sim \mathrm{Gamma}(\alpha, \lambda)$ and $Y \sim \mathrm{Gamma}(\beta, \lambda)$ are independent random variables, then $X + Y \sim \mathrm{Gamma}(\alpha + \beta, \lambda)$.

Solution

- Choose $z > 0$. Note that $f_X(x) > 0$ for $x > 0$ and $f_Y(z - x) > 0$ for $x < z$.

$$f_{X+Y}(z) = \int_0^z \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \frac{\lambda^\beta (z-x)^{\beta-1} e^{-\lambda(z-x)}}{\Gamma(\beta)} \mathrm{d}x$$

$$\overset{(a)}{=} \frac{\lambda^{\alpha+\beta} z^{\alpha+\beta-1} e^{-\lambda z}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^{\alpha-1}(1-y)^{\beta-1} \mathrm{d}y$$

Sum of Continuous Random Variables

Solution (Continued)

- Step (a) uses $y = x/z$ substitution. Finally,

$$f_{X+Y}(z) = \frac{\lambda^{\alpha+\beta} z^{\alpha+\beta-1} e^{-\lambda z}}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha, \beta) = \frac{\lambda^{\alpha+\beta} z^{\alpha+\beta-1} e^{-\lambda z}}{\Gamma(\alpha+\beta)}$$

Note

- Recall that $\exp(\lambda)$ is identical to $\mathrm{Gamma}(1, \lambda)$. Therefore, if $X_i \sim \exp(\lambda)$, $i \in \{1, \cdots, n\}$ are independent, then

$$X_1 + \cdots + X_n \sim \mathrm{Gamma}(n, \lambda)$$

Product of Random Variables

Example

Let X_1, X_2 be independent log-normal random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), respectively. Find the PDF of $X_1 X_2$.

Solution

- Note that $Y_n = \ln(X_n) \sim \mathcal{N}(\mu_n, \sigma_n^2)$, where $n \in \{1, 2\}$.
- Since Y_1, Y_2 are independent, $Y_1 + Y_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.
- Observe that $Y_1 + Y_2 = \ln(X_1 X_2)$.
- Hence, $X_1 X_2$ is log-normal with parameters $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Product of Random Variables

Proposition 12.4

If X, Y are positive random variables with joint PDF $f_{X,Y}$, then the PDF of XY is given as:

$$f_{XY}(z) = \int_0^{\infty} \frac{1}{x} f_{X,Y}\left(x, \frac{z}{x}\right) \mathrm{d}x, \quad z > 0$$

Proof of Proposition 12.4

- Choose $z > 0$. Note that

$$F_{XY}(z) = \int_0^{\infty} \int_0^{\frac{z}{x}} f_{X,Y}(x, y) \mathrm{d}y \mathrm{d}x$$

- Differentiating with respect to z, we get the result.

Product of Random Variables

Example

Let X_1, X_2 be independent Pareto random variables with parameters (a_1, λ_1) and (a_2, λ_2), respectively. Find the PDF of $X_1 X_2$.

Solution

- Note that $f_{X_1X_2}(z) = 0$ if $z < a_1a_2$. Choose $z \geq a_1a_2$. Note that

$$f_{X_1X_2}(z) = \int_{a_1}^{\frac{z}{a_2}} \frac{1}{x} \lambda_1 a_1^{\lambda_1} x^{-(\lambda_1+1)} \lambda_2 a_2^{\lambda_2} \left(\frac{z}{x}\right)^{-(\lambda_2+1)} \mathrm{d}x$$

$$= \lambda_1 \lambda_2 a_1^{\lambda_1} a_2^{\lambda_2} z^{-(\lambda_2+1)} \int_{a_1}^{\frac{z}{a_2}} x^{(\lambda_2-\lambda_1-1)} \mathrm{d}x$$

Product of Random Variables

Solution (Continued)

- If $\lambda_1 \neq \lambda_2$, then one arrives at the following.

$$f_{X_1X_2}(z) = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} a_1^{\lambda_1} a_2^{\lambda_2} z^{-(\lambda_2+1)} \left\{ \left(\frac{z}{a_2}\right)^{\lambda_2-\lambda_1} - a_1^{\lambda_2-\lambda_1} \right\}$$

$$= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} z^{-1} \left\{ \left(\frac{z}{a_1a_2}\right)^{-\lambda_1} - \left(\frac{z}{a_1a_2}\right)^{-\lambda_2} \right\}$$

- If $\lambda_1 = \lambda_2$, then the following holds.

$$f_{X_1X_2}(z) = \lambda_1\lambda_2 a_1^{\lambda_1} a_2^{\lambda_2} z^{-(\lambda_2+1)} \ln\left(\frac{z}{a_1a_2}\right)$$

Ratio of Random Variables

Example

Let X_1, X_2 be independent log-normal random variables with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), respectively. Find the PDF of X_1/X_2.

Solution

- Note that $Y_n = \ln(X_n) \sim \mathcal{N}(\mu_n, \sigma_n^2)$, where $n \in \{1, 2\}$.
- Clearly, $-Y_2 \sim \mathcal{N}(-\mu_2, \sigma_2^2)$.
- Since Y_1, Y_2 are independent, $Y_1 - Y_2 = \ln(X_1/X_2) \sim \mathcal{N}(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$.
- Therefore, X_1/X_2 is log-normal with $(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$ parameters.

Ratio of Random Variables

Proposition 12.5

If X, Y are positive random variables with joint PDF $f_{X,Y}$, then the PDF of X/Y is given as:

$$f_{X/Y}(z) = \int_0^\infty y f_{X,Y}(zy, y)\mathrm{d}y, \ z > 0$$

Proof of Proposition 12.5

- Choose $z > 0$. Differentiating the following CDF, we get our desired result.

$$F_{X/Y}(z) = \int_0^\infty \int_0^{zy} f_{X,Y}(x, y)\mathrm{d}x\mathrm{d}y$$

Ratio of Random Variables

Example

Let X, Y be independent uniform random variables in $(0, 1)$. Find the PDF of X/Y.

Solution

- Note that $f_{X/Y}(z) = 0$ if $z \leq 0$. Choose $z > 0$. We have the following.

$$f_{X/Y}(z) = \int_0^{\min\{1, \frac{1}{z}\}} y dy = \begin{cases} \frac{1}{2} & \text{if } 0 < z < 1 \\ \frac{1}{2z^2} & \text{if } 1 \leq z < \infty \end{cases}$$