Probability and Stochastic Processes
Lecture 14: Functions of Random Vectors

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Introduction

Proposition 14.1

Let (X_1, X_2) and (Y_1, Y_2) be jointly continuous random variables with joint PDFs f_{X_1,X_2}, and f_{Y_1,Y_2}, respectively such that $X_1 = h_1(Y_1, Y_2)$ and $X_2 = h_2(Y_1, Y_2)$. Assume that

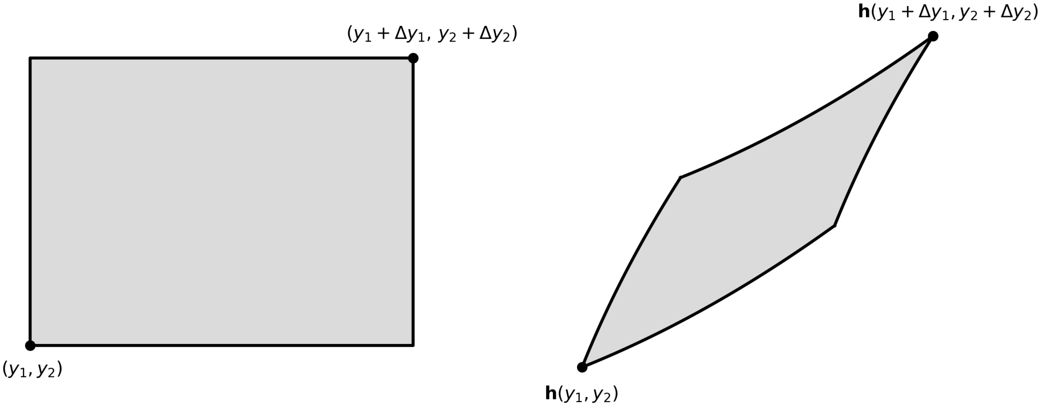
(a) for each pair (y_1, y_2), there is a unique pair (x_1, x_2), and vice versa.

(b) the functions h_1 and h_2 have continuous partial derivatives such that $\forall (y_1, y_2)$

$$J(y_1, y_2) = \begin{vmatrix} \dfrac{\partial h_1(y_1, y_2)}{\partial y_1} & \dfrac{\partial h_1(y_1, y_2)}{\partial y_2} \\ \dfrac{\partial h_2(y_1, y_2)}{\partial y_1} & \dfrac{\partial h_2(y_1, y_2)}{\partial y_2} \end{vmatrix} \neq 0$$

The following holds: $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(h_1(y_1, y_2), h_2(y_1, y_2))|J(y_1, y_2)|, \forall (y_1, y_2)$.

Introduction



Introduction

Proof Intuition of Proposition 14.1

- If Δy_1 and Δy_2 are small, then

$$\mathbf{P}((Y_1, Y_2) \in [y_1, y_1 + \Delta y_1] \times [y_2, y_2 + \Delta y_2]) \approx f_{Y_1,Y_2}(y_1, y_2)\Delta y_1 \Delta y_2$$

- $(Y_1, Y_2) \in [y_1, y_1 + \Delta y_1] \times [y_2, y_2 + \Delta y_2]$ is the same event as (X_1, X_2) lying in the (approximate) parallelogram, whose side vectors are (a_1, a_2) and (b_1, b_2), where:

$$a_j = h_j(y_1 + \Delta y_1, y_2) - h_j(y_1, y_2) \approx \frac{\partial h_j(y_1, y_2)}{\partial y_1}\Delta y_1$$

$$b_j = h_j(y_1, y_2 + \Delta y_2) - h_j(y_1, y_2) \approx \frac{\partial h_j(y_1, y_2)}{\partial y_2}\Delta y_2, \quad j \in \{1, 2\}$$

Introduction

Proof Intuition of Proposition 14.1 (Continued)

- The area of the parallelogram is the absolute value of the following quantity.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \approx \begin{vmatrix} \frac{\partial h_1(y_1,y_2)}{\partial y_1} & \frac{\partial h_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial h_2(y_1,y_2)}{\partial y_1} & \frac{\partial h_2(y_1,y_2)}{\partial y_2} \end{vmatrix} \Delta y_1 \Delta y_2 = J(y_1,y_2)\Delta y_1 \Delta y_2$$

- We have the following relation.

$$\begin{aligned} &\mathbf{P}\left((Y_1, Y_2) \in [y_1, y_1 + \Delta y_1] \times [y_2, y_2 + \Delta y_2]\right) \\ &\approx f_{X_1,X_2}(h_1(y_1,y_2), h_2(y_1,y_2))|J(y_1,y_2)|\Delta y_1 \Delta y_2 \end{aligned}$$

Applications

Example

If $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$, express the joint PDF of Y_1, Y_2 in terms of f_{X_1,X_2}, the joint PDF of X_1, X_2. In particular, find the joint PDF of Y_1, Y_2 if X_1, X_2 are independent exponential random variables with parameters λ_1 and λ_2, respectively.

Solution

- Note that $X_1 = (Y_1 + Y_2)/2$ and $X_2 = (Y_1 - Y_2)/2$. Hence,

$$J(y_1, y_2) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

- We get $f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2} f_{X_1,X_2}(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2})$.

Applications

Solution (Continued)

- Since X_1 and X_2 are independent, the joint PDF of (X_1, X_2) is as follows.

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \lambda_1\lambda_2 \exp\left(-\lambda_1 x_1 - \lambda_2 x_2\right) & \text{if } x_1, x_2 \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

- Hence, the joint PDF of (Y_1, Y_2) can be expressed as follows.

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{\lambda_1\lambda_2}{2} e^{-\lambda_1\left(\frac{y_1+y_2}{2}\right)-\lambda_2\left(\frac{y_1-y_2}{2}\right)} & \text{if } y_1+y_2, y_1-y_2 \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

- Note that Y_1 and Y_2 are NOT independent, although X_1 and X_2 are.

Applications

Example

If $X_1, X_2 \sim \mathcal{N}(0,1)$ are independent, find the joint PDF of $(X_1 + X_2, X_1 - X_2)$.

Solution

- Since X_1, X_2 are independent, the joint PDF of (X_1, X_2) is:

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{x_1^2 + x_2^2}{2}\right), \ \forall (x_1, x_2) \in \mathbb{R}^2$$

- Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. We have the following $\forall (y_1, y_2) \in \mathbb{R}^2$.

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2} f_{X_1,X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right)$$

Applications

Solution (Continued)

- Substituting the expression of f_{X_1,X_2}, we arrive at the following.

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{4\pi}\exp\left(-\frac{(y_1+y_2)^2+(y_1-y_2)^2}{8}\right) = \frac{1}{4\pi}\exp\left(-\frac{y_1^2+y_2^2}{4}\right)$$

- Hence $Y_1, Y_2 \sim \mathcal{N}(0,2)$ and they are independent.

Note

- After a transformation, an independent random vector (X_1, X_2) may or may not remain independent.

Applications

Example

Let $X, Y \sim \mathcal{N}(0,1)$ be independent. If (R, Θ) is the polar coordinate transformation of (X, Y), then find the joint PDF of (R, Θ).

Solution

- We have $X = R\cos\Theta$ and $Y = R\sin\Theta$. Hence, the Jacobian is:

$$J(r,\theta) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

Applications

Solution (Continued)

- Since X, Y are independent, their joint PDF is given below.

$$f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right), \quad \forall (x,y) \in \mathbb{R}^2$$

- The joint PDF of (R, Θ) is as follows $\forall r \geq 0$ and $\forall \theta \in (0, 2\pi)$.

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi} \exp\left(-\frac{(r\cos\theta)^2 + (r\sin\theta)^2}{2}\right) = \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right)$$

- Hence, R, Θ are independent, and R is Rayleigh distributed and Θ is uniformly distributed in $(0, 2\pi)$.

Applications

Example

Let $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta, \lambda)$ be independent. Find the joint PDF of $U = X + Y$ and $V = X/(X + Y)$.

Solution

- Since (X, Y) are independent, the joint PDF of X, Y is:

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \dfrac{\lambda^\beta y^{\beta-1} e^{-\lambda y}}{\Gamma(\beta)} & \text{if } x, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

- Note that $X = UV$ and $Y = U(1 - V)$. The Jacobian is given below.

Applications

Solution (Continued)

$$J(u,v) = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$$

- The joint density function of (U, V) is expressed below.

$$\begin{aligned} f_{U,V}(u,v) &= |u| f_{X,Y}(uv, u(1-v)) \\ &= \begin{cases} \dfrac{\lambda^{\alpha+\beta} u^{\alpha+\beta-1} e^{-\lambda u}}{\Gamma(\alpha+\beta)} \times \dfrac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha,\beta)} & \text{if } u \geq 0, v \in (0,1) \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

- Hence, U, V are independent, and $U \sim \mathrm{Gamma}(\alpha+\beta, \lambda)$ and $V \sim \mathrm{Beta}(\alpha, \beta)$.

Applications

Example

Let $X_1, \cdots, X_n$ be independent and identically distributed exponential random variables with the same parameter λ.

(a) Find the joint PDF of $(Y_1, \cdots, Y_n)$, where $Y_i = X_1 + \cdots + X_i$, $\forall i \in \{1, \cdots, n\}$.

(b) Find the PDF of Y_n.

Solution

- Note that $X_1 = Y_1, X_2 = Y_2 - Y_1, \cdots, X_n = Y_n - Y_{n-1}$.

Applications

Solution (Continued)

- The Jacobian can be computed as follows.

$$J(y_1, \cdots, y_n) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix} = 1, \quad \forall (y_1, \cdots, y_n) \in \mathbb{R}^n$$

- The joint PDF of $(Y_1, \cdots, Y_n)$ is given as follows.

Applications

Solution (Continued)

$$\begin{aligned} f_{Y_1,\cdots,Y_n}(y_1,\cdots,y_n) &= f_{X_1,\cdots,X_n}(y_1, y_2-y_1,\cdots,y_n-y_{n-1}) \\ &= f_{X_1}(y_1) f_{X_2}(y_2-y_1)\cdots f_{X_n}(y_n-y_{n-1}) \\ &= \lambda^n \exp\left(-\lambda y_n\right) \mathbf{1}(0 \le y_1 \le y_2 \le \cdots \le y_n) \end{aligned}$$

- The joint PDF of $(Y_2,\cdots,Y_n)$ is given as:

$$\begin{aligned} f_{Y_2,\cdots,Y_n}(y_2,\cdots,y_n) &= \int_0^{y_2} f_{Y_1,\cdots,Y_n}(y_1,\cdots,y_n)\mathrm{d}y_1 \\ &= \lambda^n y_2 \exp\left(-\lambda y_n\right) \mathbf{1}(0 \le y_2 \le \cdots \le y_n) \end{aligned}$$

Applications

Solution (Continued)

- Similarly, the joint PDF of $(Y_3, \cdots, Y_n)$ is given as follows.

$$\begin{aligned} f_{Y_3,\cdots,Y_n}(y_3, \cdots, y_n) &= \int_0^{y_3} f_{Y_2,Y_3,\cdots,Y_n}(y_2, y_3, \cdots, y_n)\mathrm{d}y_2 \\ &= \frac{\lambda^n y_3^2}{2!} \exp\left(-\lambda y_n\right) \mathbf{1}(0 \le y_3 \le \cdots \le y_n) \end{aligned}$$

- Repeating the process n times, we obtain:

$$f_{Y_n}(y_n) = \frac{\lambda^n y_n^{n-1}}{(n-1)!} \exp(-\lambda y_n)\mathbf{1}(y_n \ge 0)$$