Probability and Stochastic Processes
Lecture 17: Sequence of Random Variables

Prof. Washim Uddin Mondal
Department of Electrical Engineering
Indian Institute of Technology Kanpur

Sequence of Real Numbers

Definition

- A sequence of $\mathbb{R}$ is an *ordered* collection of real numbers, possibly with repetition.
- Formally, a sequence of a set A is a function of the form $f : \mathbb{N} \to A$.
- A sequence is typically denoted as $(x_n)_{n\in\mathbb{N}}$ or $(x_1, x_2, \cdots)$.

Example

- $(x_n)_{n\in\mathbb{N}}$ is an arithmetic sequence if $x_n = a + (n-1)d$, $\forall n \in \mathbb{N}$.
- $(x_n)_{n\in\mathbb{N}}$ is a geometric sequence if $x_n = ar^{n-1}$, $\forall n \in \mathbb{N}$.
- $(x_n)_{n\in\mathbb{N}}$ is a harmonic sequence if $x_n \neq 0$, $\forall n \in \mathbb{N}$, and $(x_n^{-1})_{n\in\mathbb{N}}$ is arithmetic.

Sequence of Real Numbers

Convergence

A sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathbb{R}$ is said to converge to $x \in \mathbb{R}$, denoted as $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ if $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\forall n \geq N$, $|x_n - x| < \epsilon$.

Example

(a) Let $x_n = \sin(n)/\sqrt{n}$, $\forall n \in \mathbb{N}$. Note that $|x_n| \leq 1/\sqrt{N}$, $\forall n \geq N$. Hence, if $N = \lceil \epsilon^{-2} \rceil$, we have $|x_n| < \epsilon$, $\forall n \geq N$. Clearly, $x_n \to 0$.

(b) Let $x_n = (1/2)^n$, $\forall n \in \mathbb{N}$. Note that $|x_n| \leq (1/2)^N$, $\forall n \geq N$. If $N = \lceil \log_2(1/\epsilon) \rceil$, we have $|x_n| < \epsilon$, $\forall n \geq N$. Thus, $x_n \to 0$.

Sequence of Real Numbers

Divergence

A sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathbb{R}$ is said to diverge if it does not converge to any $x \in \mathbb{R}$.

Example

- The alternating sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = (-1)^n$, $\forall n \in \mathbb{N}$.
- The geometric sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = 2^n$, $\forall n \in \mathbb{N}$.
- The sinusoidal sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = \sin(3n)$, $\forall n \in \mathbb{N}$.

Convergence of Sequence of Random Variables

Pointwise Convergence

A sequence of random variables $(X_1, X_2, \cdots)$ defined on the measurable space $(\Omega, \mathcal{F})$ is said to converge pointwise to another random variable X, also defined on the measurable space $(\Omega, \mathcal{F})$, if $\lim_{n\to\infty} X_n(\omega) = X(\omega), \forall \omega \in \Omega$.

Example

Let $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$. Find the pointwise limit of a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables, defined on the measurable space $(\Omega, \mathcal{F})$, such that

$$X_n(\omega) = \begin{cases} 0 & \text{if } \omega \in \left[0, \frac{1}{n}\right] \\ 1 & \text{if } \omega \in \left(\frac{1}{n}, 1\right] \end{cases}$$

Convergence of Sequence of Random Variables

Solution

- If $\omega = 0$, then $X_n(\omega) = 0$, $\forall n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} X_n(0) = 0$.
- If $\omega \in (0, 1]$, then $\forall n > 1/\omega$, we have $X_n(\omega) = 1$. Thus, $\lim_{n\to\infty} X_n(\omega) = 1$.
- The pointwise limit of $(X_n)_{n\in\mathbb{N}}$, denoted as X, is expressed as follows.

$$X(\omega) = \begin{cases} 0 & \text{if } \omega = 0 \\ 1 & \text{if } \omega \in (0, 1] \end{cases}$$

Note

Pointwise convergence is also called sure or certain convergence.

Convergence of Sequence of Random Variables

Almost Sure Convergence

A sequence of random variables $(X_1, X_2, \cdots)$, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, is said to almost surely converge to another random variable X, also defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, if the following equality holds.

$$\mathbf{P}\left(\left\{\omega \in \Omega \,\middle|\, \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1$$

Note

Almost sure convergence is also called convergence with probability 1.

Convergence of Sequence of Random Variables

Example

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $\mathbf{P} : \mathcal{F} \to [0,1]$ be such that $\mathbf{P}([a,b]) = b - a$, $0 \leq a \leq b \leq 1$. Consider a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that

$$X_n(\omega) = \begin{cases} 2^{-n} & \text{if } \omega \in \mathbb{Q} \cap [0,1] \\ 1 & \text{if } \omega \in [0,1] \setminus \mathbb{Q} \end{cases}$$

Prove that $(X_n)_{n\in\mathbb{N}}$ converges almost surely to a random variable X, defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that $X(\omega) = 1$, $\forall \omega \in [0,1]$.

Convergence of Sequence of Random Variables

Solution

- The pointwise limit of $(X_n)_{n\in\mathbb{N}}$ is given as follows.

$$\lim_{n\to\infty} X_n(\omega) = \begin{cases} 0 & \text{if } \omega \in \mathbb{Q}\cap[0,1] \\ 1 & \text{if } \omega \in [0,1]\setminus\mathbb{Q} \end{cases}$$

- Define $A = \{\omega \in \Omega | \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = [0,1]\setminus\mathbb{Q}$.
- Therefore, $\mathbf{P}(A) = 1 - \mathbf{P}([0,1]\cap\mathbb{Q}) = 1$. The last equality is due to countable additivity of $\mathbf{P}$, and the fact that $\mathbf{P}(\{a\}) = 0, \forall a \in [0,1]$.

Convergence of Sequence of Random Variables

Convergence in Probability

A sequence of random variables $(X_1, X_2, \cdots)$, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, is said to converge in probability to another random variable X, also defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, if the following holds $\forall \epsilon > 0$.

$$\lim_{n \to \infty} \mathbf{P}\left(|X_n - X| > \epsilon\right) = 0$$

- Note that the event $\{|X_n - X| > \epsilon\}$ is the same as the following.

$$\left\{\omega \in \Omega \middle| |X_n(\omega) - X(\omega)| > \epsilon\right\}$$

Convergence of Sequence of Random Variables

Example

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, where $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, and $\mathbf{P} : \mathcal{F} \to [0, 1]$ be such that $\mathbf{P}([a, b]) = b^2 - a^2$, $0 \le a \le b \le 1$. Consider a sequence of random variables $(X_n)_{n \in \mathbb{N}}$, defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that

$$X_n(\omega) = \begin{cases} n^2 & \text{if } \omega \in \left[0, \frac{1}{n}\right] \\ 0 & \text{if } \omega \in \left(\frac{1}{n}, 1\right] \end{cases}$$

Show that the sequence $(X_n)_{n \in \mathbb{N}}$ converges in probability to another random variable X, defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that $X(\omega) = 0$, $\forall \omega \in [0, 1]$.

Convergence of Sequence of Random Variables

Solution

- Note the following characterization.

$$|X_n(\omega) - X(\omega)| = \begin{cases} n^2 & \text{if } \omega \in \left[0, \frac{1}{n}\right] \\ 0 & \text{if } \omega \in \left(\frac{1}{n}, 1\right] \end{cases}$$

- Given $\epsilon > 0$, and $n > \sqrt{\epsilon}$, we have the following deviation probability.

$$\mathbf{P}\left(|X_n - X| > \epsilon\right) = \mathbf{P}\left(\left[0, \frac{1}{n}\right]\right) = \frac{1}{n^2}$$

- Hence, $\lim_{n\to\infty} \mathbf{P}(|X_n - X| > \epsilon) = 0$.

Convergence of Sequence of Random Variables

Convergence in Expectation

A sequence of random variables $(X_1, X_2, \cdots)$, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, is said to converge in expectation to another random variable X, also defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, if the following holds.

$$\lim_{n \to \infty} \mathbf{E}\left[|X_n - X|\right] = 0$$

Notes

- The above convergence is also called the L^1 convergence.
- We can similarly define the L^p convergence, where $p > 0$.

Convergence of Sequence of Random Variables

Example

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables, defined on the same sample space $(\Omega, \mathcal{F}, \mathbf{P})$. Define $\bar{X}_n = (X_1 + \cdots + X_n)/n$ as the nth sample mean, $\forall n \in \mathbb{N}$. If $\mathbf{E}[X_n] = \mu < \infty$ and $\mathrm{Var}(X_n) = \sigma^2 < \infty$, $\forall n \in \mathbb{N}$, then show that the sequence $(\bar{X}_n)_{n\in\mathbb{N}}$ converges in expectation to another random variables X, defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that $X(\omega) = \mu$, $\forall \omega \in \Omega$.

Solution

- $\mathbf{E}[|\bar{X}_n - X|] = \mathbf{E}[|\bar{X}_n - \mu|] \leq \left(\mathbf{E}[|\bar{X}_n - \mu|^2]\right)^{1/2} = \sigma/\sqrt{n}$.

Convergence of Sequence of Random Variables

Convergence in Distribution

A sequence of random variables $(X_1, X_2, \cdots)$, with corresponding CDFs $F_{X_1}, F_{X_2}, \cdots$ is said to converge in distribution to another random variable X, with F_X as its corresponding CDF, if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$, for all x where $F_X(\cdot)$ is continuous.

Example

Let $U \sim \text{Unif}(0, 1)$. Define $X_n = (-1)^n U/n$, $\forall n \in \mathbb{N}$. Show that the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution to another random variable X whose CDF is:

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Convergence of Sequence of Random Variables

Solution

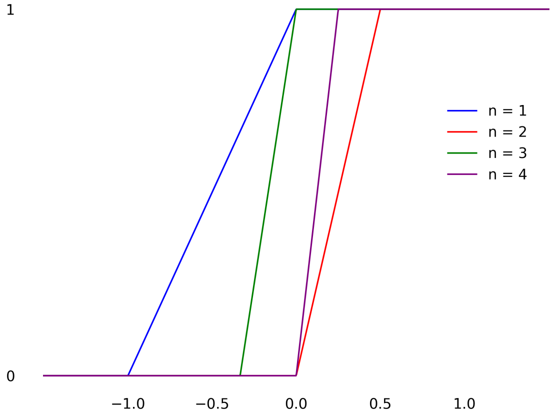
- If n is even, then the CDF of X_n is:

$$F_{X_n}(x) = \mathbf{P}(U \leq nx) = \begin{cases} 0 & \text{if } x < 0 \\ nx & \text{if } 0 \leq x < \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} \leq x \end{cases}$$

- If n is odd, then the CDF of X_n is:

$$F_{X_n}(x) = 1 - \mathbf{P}(U < -nx) = \begin{cases} 0 & \text{if } x < -\frac{1}{n} \\ 1 + nx & \text{if } -\frac{1}{n} \leq x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Convergence of Sequence of Random Variables



Convergence of Sequence of Random Variables

Solution (Continued)

- The limit of the sequence $(F_{X_n})_{n\in\mathbb{N}}$ is as follows:

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ \text{does not exist} & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

- Clearly, $\lim_{n\to\infty} F_{X_n}(x) = F_X(x), \forall x \in \mathbb{R} \setminus \{0\}$.
- Since $F_X(\cdot)$ is discontinuous at $x = 0$, the result is proved.

Convergence of Sequence of Random Variables

Notations

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and X be another random variable, also defined on $(\Omega, \mathcal{F}, \mathbf{P})$. We use the notation

- $X_n \xrightarrow{p.w.} X$ to denote that $(X_n)_{n\in\mathbb{N}}$ converges pointwise to X.
- $X_n \xrightarrow{a.s.} X$ to denote that $(X_n)_{n\in\mathbb{N}}$ converges almost surely to X.
- $X_n \xrightarrow{P} X$ to denote that $(X_n)_{n\in\mathbb{N}}$ converges in probability to X.
- $X_n \xrightarrow{E} X$ to denote that $(X_n)_{n\in\mathbb{N}}$ converges in expectation to X.
- $X_n \xrightarrow{D} X$ to denote that $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X.

Hierarchy Theorem

Theorem 17.1

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and X be another random variable, also defined on $(\Omega, \mathcal{F}, \mathbf{P})$. The following results hold.

(a) $X_n \xrightarrow{p.w.} X \implies X_n \xrightarrow{a.s.} X.$

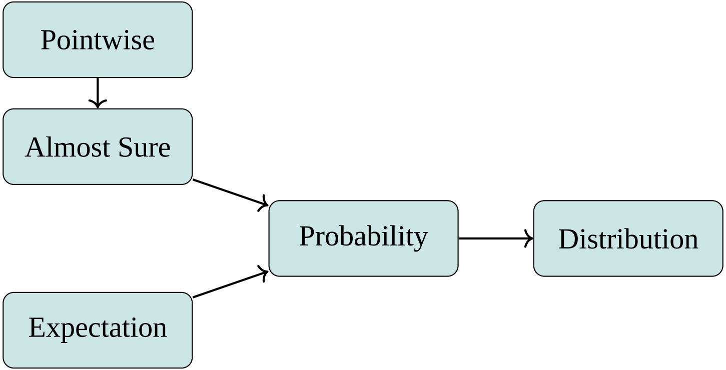
(b) $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X.$

(c) $X_n \xrightarrow{E} X \implies X_n \xrightarrow{P} X.$

(d) $X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X.$

Moreover, no other implication holds in general.

Hierarchy Theorem



Hierarchy Theorem

Proof of Part (a): Pointwise convergence implies Almost Sure Convergence

- If $X_n \xrightarrow{p.w.} X$, then $\lim_{n\to\infty} X_n(\omega) = X(\omega), \forall \omega \in \Omega$. Hence,

$$\mathbf{P}\left(\left\{\omega \in \Omega \middle| \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = \mathbf{P}\left(\Omega\right) = 1$$

- Therefore, $X_n \xrightarrow{a.s.} X$, as desired.
- Proofs of other parts are given as exercises.