Probability and Stochastic Processes
Lecture 2: Functions and Countability

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Basic Definitions

Relations

A relation from a set A to another set B is a subset of $A \times B$.

Examples

If $A = \{a, b, c\}$ and $B = \{x, y\}$, then

(a) $\{(a, x), (b, y)\}$ is a valid relation from A to B.

(b) $\{(x, a), (b, y)\}$ is not a valid relation from A to B.

Exercise

Let $|A| = p$ and $|B| = q$. Find the number of relations from A to B.

Basic Definitions

Notes

If $R \subset A \times B$ is a valid relation from A to B, then

(a) there may exist $a \in A$ and $b_1, b_2 \in B$ such that $(a, b_1) \in R$ and $(a, b_2) \in R$, i.e., one element of A might be related to multiple elements of B.

(b) there may exist $a_1, a_2 \in A$ and $b \in B$ such that $(a_1, b) \in R$ and $(a_2, b) \in R$, i.e., multiple elements of A might be related to a single element of B.

(c) there may exist $a \in A$ such that $(a, b) \notin R$, $\forall b \in B$, i.e., some elements of A might not be related to any element of B.

(d) there may exist $b \in B$ such that $(a, b) \notin R$, $\forall a \in A$, i.e., there might exist some elements of B such that no element of A is related to them.

Basic Definitions

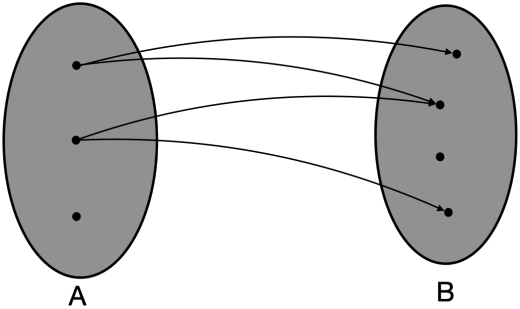


Figure: Pictorial depiction of a valid relation from A to B.

Basic Definitions

Functions

A function from A to B, denoted as $f : A \to B$, is a relation that maps each element of A to an element of B. The sets A and B are called the domain and co-domain of f, respectively.

Example

If $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $f \subset A \times B$ is such that

(a) $f = \{(a, 1), (b, 1), (c, 2)\}$, then f is a valid function from A to B.

(b) $f = \{(a, 1), (a, 2), (b, 3), (c, 4)\}$, then f is NOT a valid function from A to B.

(c) $f = \{(a, 1), (b, 3)\}$, then f is NOT a valid function from A to B.

Basic Definitions

Exercise

Find the number of functions from A to B if $|A| = p$ and $|B| = q$.

Notes

If $f \subset A \times B$ is a valid function from A to B, then

(a) every element of A is mapped to some element of B.

(b) no element of A is mapped to multiple elements of B.

(c) multiple element of A may get mapped to a single element in B.

(d) some element of B may be such that no element of A is mapped to them.

Basic Definitions

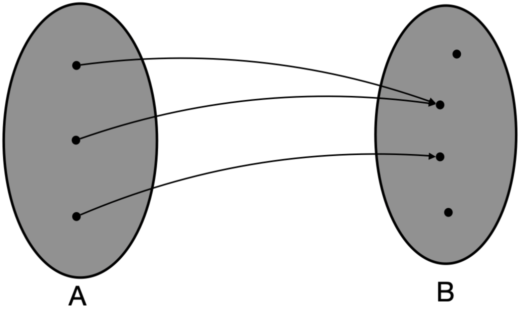


Figure: Pictorial depiction of a valid function from A to B.

Basic Definitions

Pre-image

If $f : A \to B$ is a function, and $C \subset B$, then the pre-image of C is defined as

$$f^{-1}(C) = \{x \in A \mid f(x) \in C\}$$

- If $C = \{c\}$, then $f^{-1}(\{c\}) = \{x \in A \mid f(x) = c\}$.
- $f^{-1}(\phi) = \{x \in A \mid f(x) \in \phi\} = \phi$, i.e., the pre-image of an empty set is an empty set.
- $f^{-1}(B) = \{x \in A \mid f(x) \in B\} = A$, i.e., the pre-image of the co-domain is the domain.

Range

If $f : A \to B$ is a function, then $\mathrm{range}(f) = \{f(x) \mid x \in A\}$.

Basic Definitions

Examples

If $f : [0,1] \to [0,2]$ is such that $f(x) = x^2, \forall x \in [0,1]$, then

(a) $f^{-1}([0, 0.25]) = [0, 0.5]$

(b) $\mathrm{range}(f) = [0, 1]$

Notes

If $f : A \to B$ is a function, then

(a) $\mathrm{range}(f) \subset B$ and the equality holds if every $b \in B$ has a non-empty pre-image.

(b) $f^{-1}(\mathrm{range}(f)) = A$, i.e., the pre-image of the range is the domain.

Properties of Pre-images

Proposition 2.1

If $f : A \to B$ is a function, and $C_\alpha \subset B, \forall \alpha \in I$, then

(a) $f^{-1}(C_\alpha^c) = [f^{-1}(C_\alpha)]^c, \forall \alpha \in I$.

(b) $f^{-1}(\cup_{\alpha \in I} C_\alpha) = \cup_{\alpha \in I} f^{-1}(C_\alpha)$.

(c) $f^{-1}(\cap_{\alpha \in I} C_\alpha) = \cap_{\alpha \in I} f^{-1}(C_\alpha)$.

Proof of Proposition 2.1(a)

Note the following implications.

$$x \in f^{-1}(C_\alpha^c) \iff f(x) \in C_\alpha^c \iff f(x) \notin C_\alpha \iff x \notin f^{-1}(C_\alpha) \iff x \in [f^{-1}(C_\alpha)]^c$$

Properties of Pre-images

Proof of Proposition 2.1(b)

$$x \in f^{-1}(\cup_{\alpha \in I} C_\alpha) \iff f(x) \in \cup_{\alpha \in I} C_\alpha$$
$$\iff f(x) \in C_\alpha \text{ for some } \alpha \in I$$
$$\iff x \in f^{-1}(C_\alpha) \text{ for some } \alpha \in I \iff x \in \cup_{\alpha \in I} f^{-1}(C_\alpha)$$

Proof of Proposition 2.1(c)

$$x \in f^{-1}(\cap_{\alpha \in I} C_\alpha) \iff f(x) \in \cap_{\alpha \in I} C_\alpha$$
$$\iff f(x) \in C_\alpha \text{ for all } \alpha \in I$$
$$\iff x \in f^{-1}(C_\alpha) \text{ for all } \alpha \in I \iff x \in \cap_{\alpha \in I} f^{-1}(C_\alpha)$$

Types of Functions

Injective Functions

A function $f : A \to B$ is called injective if every $a \in A$ gets mapped to a unique $b \in B$, i.e.,

$$f(a) = f(b) \iff a = b$$

For example, if $f : [0, 1] \to [0, 1]$ is such that

(a) $f(x) = x^2, \forall x \in [0, 1]$, then f is injective.

(b) $f(x) = \sin(\pi x), \forall x \in [0, 1]$, then f is NOT injective.

Exercise

Find the number of injective functions from A to B if $|A| = p$, $|B| = q$ and $p \leq q$.

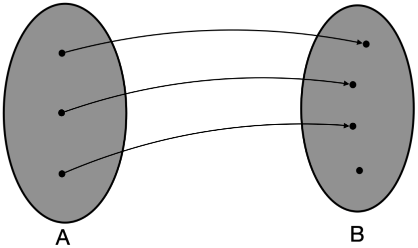
Types of Functions

Notes

If $f : A \to B$ is injective, then

(a) some $b \in B$ may have an empty pre-image i.e., $f^{-1}(\{b\}) = \phi$.

(b) $|A| \leq |B|$ if A, B are finite sets.



Types of Functions

Surjective Functions

A function $f : A \to B$ is called surjective if every $b \in B$ has a non-empty pre-image, i.e., $\text{range}(f) = B$. For example,

(a) if $f : [0, 1] \to [0, 2]$ is such that $f(x) = \sin(\pi x)$, $\forall x \in [0, 1]$, then f is NOT a surjection.

(b) if $f : [0, 1] \to [0, 2]$ is such that $f(x) = 2x$, $\forall x \in [0, 1]$, then f is a surjection.

Notes

If $f : A \to B$ is a surjection, then

(a) f may not be an injection.

(b) $|A| \geq |B|$ if A, B are finite sets.

Types of Functions

Bijective Functions

A function $f : A \to B$ is called bijective if it is both injective and surjective.

For example, if $f : [0, 1] \to [0, 2]$ is such that

(a) $f(x) = 2x$, $\forall x \in [0, 1]$, then f is a bijection.

(b) $f(x) = x$, $\forall x \in [0, 1]$, then f is an injection but not a surjection.

(c) $f(x) = 2\sin(\pi x)$, $\forall x \in [0, 1]$, then f is a surjection but not an injection.

Note

If $f : A \to B$ is a bijection, and A, B are finite sets, then $|A| = |B|$.

Cardinality of Infinite Sets

Motivation

(a) Infinite sets, by definition, have an infinite number of elements.

(b) Some of these infinities might be larger than others. Some might be the same.

Comparative Cardinality

Let A, B be two sets (either finite or infinite). We say

(a) $|A| = |B|$ if there is a bijection $f : A \to B$.

(b) $|A| \leq |B|$ if there is an injection $f : A \to B$.

(c) $|A| < |B|$ if there is an injection $f : A \to B$ but no bijection $g : A \to B$.

Cardinality of Infinite Sets

Examples

(a) Let $A = [0,1]$, $B = [0,2]$. We can define a bijection $f : A \to B$ such that $f(x) = 2x$, $\forall x \in [0,1]$. Hence $|A| = |B|$, although A is a proper subset of B.

(b) Let $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $2\mathbb{N} = \{2, 4, 6, \cdots\}$. We can define a bijection $f : \mathbb{N} \to 2\mathbb{N}$ as $f(n) = 2n$, $\forall n \in \mathbb{N}$. Hence, $|\mathbb{N}| = |2\mathbb{N}|$.

(c) Let $A = \mathbb{N}$ and $B = 2^{\mathbb{N}}$. We can define $f : A \to B$ as $f(n) = \{n\}$, $\forall n \in \mathbb{N}$. Clearly, f is an injection. Therefore, $|A| \leq |B|$.

Cardinality of Infinite Sets

Schröder-Bernstein Theorem

If there is an injection $f : A \to B$ and an injection $g : B \to A$, then there exists a bijection $h : A \to B$. In other words, $|A| \leq |B|$ and $|B| \leq |A|$ implies $|A| = |B|$.

Example

Let $A = \mathbb{N}$ and $B = \mathbb{N} \times \mathbb{N}$. Finding a bijection between A and B directly might be difficult. We can, however, apply the Schröder-Bernstein Theorem.

- Define an injection $f : A \to B$ such that $f(n) = (n, n)$.
- Define an injection $f : B \to A$ such that $f(p, q) = 2^p 3^q$.
- Therefore, $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

Cardinality of Infinite Sets

Proposition 2.2

There exists no bijection between any set A and its power set 2^A.

Proof by Contradiction

Let $f : A \to 2^A$ be a bijection. Define the following set.

$$B = \{a \in A \mid a \notin f(a)\}$$

- Since $B \subset A$, it follows that $B \in 2^A$.
- Since f is a bijection, $\exists a_0 \in A$ such that $f(a_0) = B$.

Cardinality of Infinite Sets

Proof by Contradiction (Continued)

- If $a_0 \in B$, then $a_0 \notin f(a_0) = B$.
- If $a_0 \notin B$, then $a_0 \in f(a_0) = B$.
- Hence, our initial assumption must be wrong, and f cannot be a bijection.

Corollary

Define an injection $f : A \to 2^A$ as $f(a) = \{a\}, \forall a \in A$. Using Proposition 2.2, we arrive at the conclusion that $|A| < |2^A|$.

Countability

Definition

A set A is said to be countable if it is either finite or there exists a bijection $f : \mathbb{N} \to A$.

Examples

From previous exercises, one can conclude the following statements.

(a) The set of even numbers $2\mathbb{N}$ is countable.

(b) The set of pair of natural numbers $\mathbb{N} \times \mathbb{N}$ is countable.

(c) The power set $2^{\mathbb{N}}$ is not countable.

Countability

Proposition 2.3

The interval set $[0, 1]$ is not countable.

Proof via Cantor's Diagonalization Argument

Let $f : \mathbb{N} \to [0, 1]$ be a function and $0.b_{k1}b_{k2}\cdots$ be the binary expansion of $f(k)$. Hence,

$$
\begin{aligned}
f(1) &= 0.b_{11}b_{12}b_{13}\cdots \\
f(2) &= 0.b_{21}b_{22}b_{23}\cdots \\
f(3) &= 0.b_{31}b_{32}b_{33}\cdots \\
&\vdots
\end{aligned}
$$

Countability

Proof via Cantor's Diagonalization Argument (Continued)

Consider the following number in $[0, 1]$.

$$0.\bar{b}_{11}\bar{b}_{22}\bar{b}_{33}\cdots$$

where $\bar{b}_{11} \neq b_{11}$, $\bar{b}_{22} \neq b_{22}$, and so on.

- Clearly, the above number is different from $f(1)$ in the 1st bit, different from $f(2)$ in the 2nd bit, and so on.
- For no $k \in \mathbb{N}$, $f(k)$ equals the above number.
- f cannot be surjective, and thereby, bijective.