Probability and Stochastic Processes
Lecture 20: Stochastic Processes-I

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Basics

Definition

A random process or a stochastic process X is an indexed collection of random variables $\{X_t\}_{t\in\mathbb{T}}$, all defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Notes

- The index set $\mathbb{T}$ is often interpreted as time, although it need not be.
- If $\mathbb{T} = \mathbb{N}, \mathbb{Z}$ or some collection of consecutive integers, then $\{X_t\}_{t\in\mathbb{T}}$ is called a discrete time random process.
- If $\mathbb{T} = \mathbb{R}$ or an interval of $\mathbb{R}$, then $\{X_t\}_{t\in\mathbb{T}}$ is called a continuous time random process.

Basics

Practical Examples

- Assume that a meteorologist measures the temperature of a given location at a certain time of the day. If X_t is the recorded temperature of the tth day, then $\{X_t\}_{t\in\mathbb{N}}$ is a stochastic process. Such processes are often called temporal processes because the index set is indicative of time.
- Assume that a base station (BS) is located at the origin in an infinite 2D plane, which transmits signals with a certain power. If S_r is the signal strength received by an user located at $r \in \mathbb{R}^2$, then $\{S_r\}_{r\in\mathbb{R}^2}$ is a stochastic process. Such processes are often called spatial processes because the index set is indicative of space.
- One can similarly define a spatio-temporal process.

Basics

Definition

If $\omega \in \Omega$, then $\{X_t(\omega)\}_{t\in\mathbb{T}}$ is called a realization of the stochastic process $\{X_t\}_{t\in\mathbb{T}}$, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

- If $\mathbb{T}$ is interpreted as the time index, then $\{X_t(\omega)\}_{t\in\mathbb{T}}$ is also called a sample path of the stochastic process $\{X_t\}_{t\in\mathbb{T}}$.

Example

Let $\{W_k\}_{k\in\mathbb{N}}$ be independent random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{P}(W_k = +1) = \mathbf{P}(W_k = -1) = 0.5$, $\forall k \in \mathbb{N}$. Let $X_k = W_1 + \cdots + W_k$, $\forall k \in \mathbb{N}$. Both $\{W_k\}_{k\in\mathbb{N}}$ and $\{X_k\}_{k\in\mathbb{N}}$ are discrete time random processes.

Basics

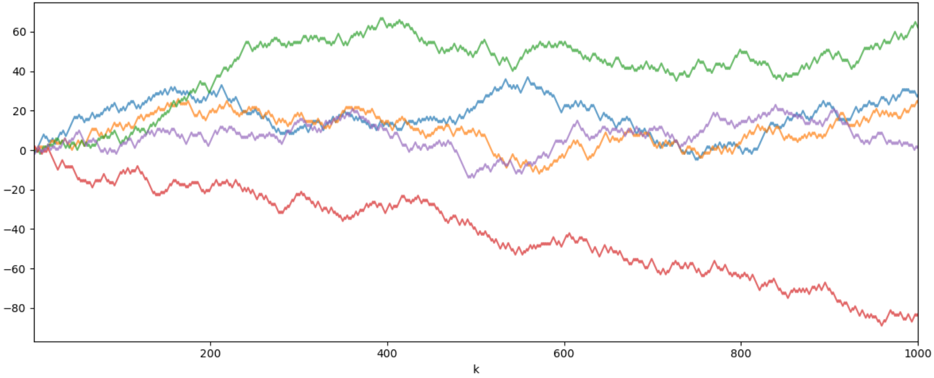

Figure: Sample paths of $\{X_k\}_{k \in \mathbb{N}}$

Basics

Definition

Let $X = \{X_t\}_{t \in \mathbb{T}}$ be a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We define the

- mean function $\mu_X : \mathbb{T} \to \mathbb{R}$ such that $\mu_X(t) = \mathbf{E}[X_t], \forall t \in \mathbb{T}$.
- (auto)-correlation function $R_X : \mathbb{T}^2 \to \mathbb{R}$ such that $R_X(s,t) = \mathbf{E}[X_s X_t], \forall (s,t) \in \mathbb{T}^2$.
- (auto)-covariance function $C_X : \mathbb{T}^2 \to \mathbb{R}$ such that $\forall (s,t) \in \mathbb{T}^2$,

$$C_X(s,t) = \mathrm{Cov}(X_s, X_t) = R_X(s,t) - \mu_X(s)\mu_X(t)$$

- nth order CDF $F_{X,n} : (\mathbb{R} \times \mathbb{T})^n \to [0,1]$ such that $\forall (x_1, t_1, \cdots, x_n, t_n) \in (\mathbb{R} \times \mathbb{T})^n$, $F_{X,n}(x_1, t_1; \cdots; x_n, t_n) = \mathbf{P}(X_{t_1} \leq x_1, \cdots, X_{t_n} \leq x_n)$.

Basics

Definition

A random process $\{X_t\}_{t\in\mathbb{T}}$ is called a second-order random process if $\mathbf{E}[X_t^2] < \infty, \forall t \in \mathbb{T}$.

- The mean, correlation, and covariance functions are well-defined for a second-order random process.

Definition

If $X = \{X_t\}_{t\in\mathbb{T}}$ is a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that X_t is a discrete random variable $\forall t \in \mathbb{T}$, then the nth order PMF of X is a function $p_{X,n} : (\mathbb{R} \times \mathbb{T})^n \to [0,1]$ such that $p_{X,n}(x_1, t_1; \cdots; x_n, t_n) = \mathbf{P}(X_{t_1} = x_1, \cdots, X_{t_n} = x_n)$, $\forall (x_1, t_1, \cdots, x_n, t_n) \in (\mathbb{R} \times \mathbb{T})^n$.

Basics

Definition

If $X = \{X_t\}_{t \in \mathbb{T}}$ is a stochastic process such that $(X_{t_1}, \cdots, X_{t_n})$ are jointly continuous random variables $\forall (t_1, \cdots, t_n) \in \mathbb{T}^n$, then the nth order PDF of X is a function $f_{X,n} : (\mathbb{R} \times \mathbb{T})^n \to [0, \infty)$ such that $f_{X,n}(x_1, t_1; \cdots; x_n, t_n) = f_{X_{t_1}, \cdots, X_{t_n}}(x_1, \cdots, x_n)$, where $f_{X_{t_1}, \cdots, X_{t_n}}$ is the joint PDF of $(X_{t_1}, \cdots, X_{t_n})$, and $(x_1, t_1, \cdots, x_n, t_n) \in (\mathbb{R} \times \mathbb{T})^n$.

Example

Let $X_t = A + Bt + t^2, \forall t \in \mathbb{R}$, where $A, B \sim \mathcal{N}(0, 1)$ are independent. Find the mean, correlation, and covariance functions, and the first-order PDF of $X = \{X_t\}_{t \in \mathbb{R}}$.

Basics

Solution

(a) The mean function is $\mu_X(t) = \mathbf{E}[X_t] = \mathbf{E}[A] + \mathbf{E}[B]t + t^2 = t^2, \forall t \in \mathbb{R}$.

(b) The correlation function is as follows $\forall (s,t) \in \mathbb{R}^2$.

$$\begin{aligned} R_X(s,t) = \mathbf{E}[X_s X_t] &= \mathbf{E}[A^2] + \mathbf{E}[B^2]st + s^2t^2 + \mathbf{E}[A]\mathbf{E}[B](s+t) \\ &+ \mathbf{E}[B]st(s+t) + \mathbf{E}[A](s^2+t^2) = 1 + st + s^2t^2 \end{aligned}$$

(c) The covariance function is as follows $\forall (s,t) \in \mathbb{R}^2$.

$$C_X(s,t) = R_X(s,t) - \mu_X(s)\mu_X(t) = 1 + st$$

Basics

Solution (Continued)

(d) Note that $X_t \sim \mathcal{N}(t^2, 1+t^2)$, $\forall t \in \mathbb{R}$. Therefore, $\forall (x,t) \in \mathbb{R}^2$,

$$f_{X,1}(x,t) = \frac{1}{\sqrt{2\pi(1+t^2)}} \exp\left(-\frac{(x-t^2)^2}{2(1+t^2)}\right)$$

Example

Let $U = \{U_k\}_{k\in\mathbb{Z}}$ be a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{P}(U_k = 1) = \mathbf{P}(U_k = -1) = 0.5$, $\forall k \in \mathbb{N}$, and U_k's are independent. Define another stochastic process $X = \{X_t\}_{t\in\mathbb{R}}$ such that $X_t = U_{\lfloor t \rfloor}$, $\forall t \in \mathbb{R}$. Find the correlation functions of U and X, the nth-order PMF of U, and the 2nd-order PMF of X.

Basics

Solution

(a) The correlation function of U is given below where $(s,t) \in \mathbb{Z}^2$.

$$R_U(s,t) = \mathbf{E}[U_s U_t] = \begin{cases} 0 & \text{if } s \neq t \\ 1 & \text{if } s = t \end{cases}$$

(b) The correlation function of X is given below where $(s,t) \in \mathbb{R}^2$.

$$R_X(s,t) = \mathbf{E}[X_s X_t] = \begin{cases} 0 & \text{if } \lfloor s \rfloor \neq \lfloor t \rfloor \\ 1 & \text{if } \lfloor s \rfloor = \lfloor t \rfloor \end{cases}$$

Basics

Solution (Continued)

(c) The nth order PMF of U is given as follows for *distinct* integers $k_1, \cdots, k_n$.

$$p_{U,n}(x_1, k_1; \cdots; x_n, k_n) = \begin{cases} 2^{-n} & \text{if } (x_1, \cdots, x_n) \in \{-1, 1\}^n \\ 0 & \text{elsewhere} \end{cases}$$

(d) The 2nd order PMF of X is given as follows.

$$p_{X,n}(x_1, t_1; x_2, t_2) = \begin{cases} 0.5 & \text{if } \lfloor t_1 \rfloor = \lfloor t_2 \rfloor,\ x_1 = x_2 \in \{-1, 1\} \\ 0.25 & \text{if } \lfloor t_1 \rfloor \neq \lfloor t_2 \rfloor,\ (x_1, x_2) \in \{-1, 1\}^2 \\ 0 & \text{elsewhere} \end{cases}$$

Random Walk

Definition

Let $X_0, W_1, W_2, \cdots$ be independent random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{P}(W_k = 1) = p = 1 - \mathbf{P}(W_k = -1)$, $\forall k \in \mathbb{N}$. Let $X_k = X_{k-1} + W_k$, $\forall k \in \mathbb{N}$. The random process $\{X_k\}_{k \in \mathbb{N} \cup \{0\}}$ is called a random walk with parameter p.

- If $p = 0.5$, the random walk is symmetric.
- If $p \neq 0.5$, the random walk is asymmetric.

Notation

- We will denote the conditional probability $\mathbf{P}(A|X_0 = k)$ as $\mathbf{P}_k(A)$ for any $A \in \mathcal{F}$. The notations $\mathbf{E}_k[\cdot]$ and $\mathrm{Var}_k[\cdot]$ are defined similarly.

Random Walk

Proposition 20.1

If $\{X_k\}_{k\in\{0,1,\cdots\}}$ is a p-parameter random walk on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, then

(a) $\mathbf{E}_k[X_n] = k + n(2p-1), \forall n \in \mathbb{N}$.

(b) $\mathrm{Var}_k[X_n] = 4np(1-p), \forall n \in \mathbb{N}$.

(c) $\mathbf{P}_k(X_n = k + j - (n-j)) = \binom{n}{j} p^j (1-p)^{n-j}, \forall n \in \mathbb{N}$.

(d) The following conditional convergences hold.

$$\mathbf{P}_k\left(\lim_{n\to\infty}\frac{X_n}{n} = 2p-1\right) = 1, \quad \lim_{n\to\infty}\mathbf{E}_k\left[\left(\frac{X_n}{n} - (2p-1)\right)^2\right] = 0$$

Random Walk

Proposition 20.1 (Continued)

(e) The following convergence in distribution holds, $\forall a \in \mathbb{R}$.

$$\lim_{n\to\infty} \mathbf{P}_k\left(\frac{X_n - n(2p-1)}{\sqrt{4np(1-p)}} \leq a\right) = \Phi(a)$$

Proof of Proposition 20.1

- Note that $\mathbf{E}[W_i] = 2p - 1, \forall i \in \mathbb{N}$.
- Moreover, $\mathrm{Var}(W_i) = 1 - (2p-1)^2 = 4p(1-p), \forall i \in \mathbb{N}$.

Proof of Proposition 20.1 (Continued)

$$\begin{aligned}\mathbf{E}_k[X_n] &= \mathbf{E}[k + W_1 + \cdots + W_n \mid X_0 = k] \\ &\overset{(a)}{=} \mathbf{E}[k + W_1 + \cdots + W_n] = k + n(2p-1)\end{aligned}$$

- Step (a) follows because X_0 is independent of $W_1, \cdots, W_n$. Similarly,

$$\begin{aligned}\mathrm{Var}_k(X_n) &= \mathrm{Var}(k + W_1 + \cdots + W_n \mid X_0 = k) \\ &= \mathrm{Var}(W_1 + \cdots + W_n) \overset{(a)}{=} \sum_{i=1}^{n} \mathrm{Var}(W_i) = 4np(1-p)\end{aligned}$$

- Step (a) follows since $W_1, \cdots, W_n$ are independent.

Random Walk

Proof of Proposition 20.1 (Continued)

- Since X_0 is independent of W_i's, we have:

$$\mathbf{P}_k\left(X_n = k + 2j - n\right) = \mathbf{P}_k\left(\sum_{i=1}^{n} W_i = 2j - n\right) = \mathbf{P}\left(\sum_{i=1}^{n} W_i = 2j - n\right)$$

- Note that $\{\sum_{i=1}^{n} W_i = 2j - n\}$ is the same as the event that j random variables in the collection $\{W_1, \cdots, W_n\}$ takes $+1$ value and the rest takes -1 value. Hence,

$$\mathbf{P}_k(X_n = k + j - (n - j)) = \binom{n}{j} p^j (1 - p)^{n-j}$$

Proof of Proposition 20.1 (Continued)

- Let $\bar{W}_n = (W_1 + \cdots + W_n)/n$. Using the strong law of large numbers, we obtain

$$\mathbf{P}_k\left(\lim_{n\to\infty} \frac{X_n}{n} = 2p-1\right) = \mathbf{P}\left(\lim_{n\to\infty} \left\{\bar{W}_n + \frac{k}{n}\right\} = 2p-1\right)$$
$$= \mathbf{P}\left(\lim_{n\to\infty} \bar{W}_n = 2p-1\right) = 1$$

- The L_2 convergence can be proven as follows. Note that

$$\mathbf{E}_k\left[\left(\frac{X_n}{n} - (2p-1)\right)^2\right] = \mathbf{E}\left[\left(\bar{W}_n - (2p-1) + \frac{k}{n}\right)^2\right]$$

Random Walk

Proof of Proposition 20.1 (Continued)

- Expanding the quadratic term, we obtain:

$$\mathbf{E}_k\left[\left(\frac{X_n}{n}-(2p-1)\right)^2\right]=\mathbf{E}\left[(\bar{W}_n-(2p-1))^2\right]+\frac{2k}{n}(\mathbf{E}[\bar{W}_n]-(2p-1))+\frac{k^2}{n^2}$$

$$\overset{(a)}{=}\mathrm{Var}(\bar{W}_n)+\frac{k^2}{n^2}\overset{(b)}{=}\frac{4p(1-p)}{n}+\frac{k^2}{n^2}$$

- Step (a) uses $\mathbf{E}[\bar{W}_n]=2p-1$, while (b) uses the fact that

$$\mathrm{Var}(\bar{W}_n)=\frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n}W_i\right)=\frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(W_i)$$

Random Walk

Proof of Proposition 20.1 (Continued)

- The convergence in distribution can be proven via the CLT. Note that

$$p_n \triangleq \mathbf{P}_k\left(\frac{X_n - n(2p-1)}{\sqrt{4np(1-p)}} \le a\right) = \mathbf{P}\left(\frac{\bar{W}_n - (2p-1)}{\sqrt{4p(1-p)/n}} \le a - \frac{k}{\sqrt{4np(1-p)}}\right)$$

- Fix $\epsilon > 0$. Note that $\forall n > k^2/(4\epsilon^2 p(1-p))$, we have the following.

$$\mathbf{P}\left(\frac{\bar{W}_n - (2p-1)}{\sqrt{4p(1-p)/n}} \le a - \epsilon\right) \le p_n \le \mathbf{P}\left(\frac{\bar{W}_n - (2p-1)}{\sqrt{4p(1-p)/n}} \le a\right)$$

- We get $\lim_{\epsilon \to 0^+} \Phi(a - \epsilon) \le \lim_{n \to \infty} p_n \le \Phi(a)$.

Gambler's Ruin

Problem

Let $X_0 = k$ be a gambler's initial wealth and X_n, $n \in \{1, 2, \cdots\}$ denote his wealth at time n. At time n, the gambler either wins or loses one unit of wealth, denoted by a random variable W_n. Assume that $\{W_n\}_{n \in \mathbb{N}}$ is a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathbf{P}(W_n = 1) = p = 1 - \mathbf{P}(W_n = -1)$, and W_n's are independent. Find the probability that the gambler accumulates $b > k$ units of wealth before being ruined, i.e., his wealth becoming zero.

Solution

- Let S be the event that the gambler accumulates b units of wealth before being ruined.

Gambler's Ruin

Solution (Continued)

- Let $p_k = \mathbf{P}(S|X_0 = k)$. Clearly, $p_0 = 0$ and $p_b = 1$. Moreover, $\forall r \in \{1, \cdots, b-1\}$,

$$\begin{aligned} p_r &= \mathbf{P}(W_1 = 1)\mathbf{P}(S|X_0 = r, W_1 = 1) + \mathbf{P}(W_1 = -1)\mathbf{P}(S|X_0 = r, W_1 = -1) \\ &= \mathbf{P}(W_1 = 1)\mathbf{P}(S|X_1 = r+1) + \mathbf{P}(W_1 = -1)\mathbf{P}(S|X_1 = r-1) \\ &= pp_{r+1} + (1-p)p_{r-1} \end{aligned}$$

- The above relation can also be written as follows $\forall r \in \{1, \cdots, b-1\}$.

$$p_{r+1} - p_r = \left(\frac{1-p}{p}\right)(p_r - p_{r-1})$$

Gambler's Run

Solution (Continued)

- Since $p_0 = 0$, we arrive at the following $\forall r \in \{0, 1, \cdots, b-1\}$.

$$p_{r+1} - p_r = \left(\frac{1-p}{p}\right)^r p_1$$

- Therefore, $\forall r \in \{1, \cdots, b\}$, we obtain the following expression.

$$p_r = \sum_{i=0}^{r-1} \left(\frac{1-p}{p}\right)^i p_1$$

Gamler's Ruin

Solution (Continued)

- Since $p_b = 1$, the final solution is given as follows.

$$p_k = \frac{\sum_{i=0}^{k-1} \left(\frac{1-p}{p}\right)^i}{\sum_{i=0}^{b-1} \left(\frac{1-p}{p}\right)^i} = \begin{cases} \frac{k}{b} & \text{if } p = \frac{1}{2} \\ \frac{1-\left(\frac{1-p}{p}\right)^k}{1-\left(\frac{1-p}{p}\right)^b} & \text{if } p \neq \frac{1}{2} \end{cases}$$