Probability and Stochastic Processes
Lecture 21: Stochastic Processes-II

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Gaussian Vector

Definition

An $\mathbb{R}^n$-valued continuous random vector X is said to be jointly Gaussian with parameters $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$, with $\boldsymbol{\Sigma}$ being positive definite, if the joint PDF of X is given as:

$$f_X(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} |\det(\boldsymbol{\Sigma})|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right), \ \forall \boldsymbol{x} \in \mathbb{R}^n$$

Notes

- Since $\boldsymbol{\Sigma}$ is positive definite, it must be symmetric by definition.
- $\boldsymbol{\Sigma}$ is guaranteed to be invertible.

Gaussian Vector

Example

Let $X = (X_1, X_2)$, $\boldsymbol{\mu} = (\mu_1, \mu_2)^\top$ and $\boldsymbol{\Sigma}$ is given as:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

where $\rho \in (-1, 1)$, and $\sigma_1, \sigma_2 > 0$. The joint PDF of X is given as follows $\forall (x_1, x_2) \in \mathbb{R}^2$.

$$f_X(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$
$$\times \exp\left(-\frac{1}{2}\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}\right)$$

Gaussian Process

Definition

A stochastic process $X = \{X_t\}_{t \in \mathbb{T}}$ is said to be a Gaussian process if for any $n \in \mathbb{N}$, and any $t_1, \cdots, t_n \in \mathbb{T}$, the random variables $X_{t_1}, \cdots, X_{t_n}$ are jointly Gaussian.

Example

If $X_n \sim \mathcal{N}(0, \sigma^2)$, $\forall n \in \mathbb{N}$, and X_n's are independent, then $\{X_n\}_{n \in \mathbb{N}}$ is a discrete time Gaussian process.

Note

The above process is the foundation of the additive white Gaussian noise (AWGN) model used in wireless networks.

Counting Process

Definition

A continuous time random process $\{N_t\}_{t\geq 0}$ is said to be a counting process if:

(a) N_t is integer-valued $\forall t \geq 0$.

(b) $N_t \geq 0$ (pointwise), $\forall t \geq 0$.

(c) If $0 \leq s < t$, then $N_s \leq N_t$ (pointwise).

Interpretation

- N_t denotes the number of events in $[0, t]$, $\forall t \geq 0$.
- For $0 \leq s < t$, $N_t - N_s$ is the number of events in $(s, t]$.

Poisson Process

Definition

A counting process $\{N_t\}_{t \geq 0}$ is said to have independent increments if the number of events occurring in disjoint intervals are independent, i.e., if $0 \leq s_1 < t_1 < \cdots < s_n < t_n$, then $N_{t_1} - N_{s_1}, \cdots, N_{t_n} - N_{s_n}$ must be independent random variables.

Definition D.1

A counting process $\{N_t\}_{t \geq 0}$ is said to be a Poisson process with rate $\lambda > 0$ if:

(a) $N_0 = 0$ (pointwise).

(b) the process $\{N_t\}_{t \geq 0}$ has independent increments.

(c) $\forall s, t \geq 0$, $N_{s+t} - N_s \sim \text{Poisson}(\lambda t)$.

Poisson Process

Notes

(a) states that the counting starts at $t = 0$.

(b) states that the numbers of events occurring at disjoint intervals are independent.

(c) states that the number of events occurring in *any* interval of length t is $\mathrm{Poisson}(\lambda t)$.

- Conditions (a) and (b) are easy to verify for a given process, while (c) may not be.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is $o(h)$ if $\lim_{h \to 0} f(h)/h = 0$.

- For example, $f(h) = h^2$ is $o(h)$ but $f(h) = h$ is not $o(h)$.

Poisson Process

Definition D.2

A counting process $\{N_t\}_{t\geq 0}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is said to be a Poisson process with rate λ if:

(a) $N_0 = 0$ (pointwise).

(b) $\{N_t\}_{t\geq 0}$ has independent increments.

(c) $\{N_t\}_{t\geq 0}$ has stationary increments, i.e., $\forall s, t \geq 0$, the distribution of $N_{t+s} - N_s$ does not depend on s.

(d) $\mathbf{P}(N_h = 1) = \lambda h + o(h)$.

(e) $\mathbf{P}(N_h \geq 2) = o(h)$.

Poisson Process

Notes

- It is easy to verify that D.1 implies D.2. In particular, if $N_h \sim \text{Poisson}(\lambda h)$, then

$$\mathbf{P}(N_h = 1) = \lambda h e^{-\lambda h} = \lambda h \left(1 - \frac{\lambda h}{1!} + \frac{(\lambda h)^2}{2!} - \cdots\right) = \lambda h + o(h)$$

- Similarly, one can show that $\mathbf{P}(N_h \geq 2) = o(h)$.
- The following proposition shows that D.2 implies D.1 as well. Therefore, these two definitions are equivalent.

Poisson Process

Proposition 21.1

Definition D.2 implies that $N_{s+t} - N_s \sim \text{Poisson}(\lambda t), \forall s, t \geq 0.$

Proof of Proposition 21.1

- Due to stationary increments, it is sufficient to prove that $N_t \sim \text{Poisson}(\lambda t), \forall t \geq 0.$
- Define $p_n(t) = \mathbf{P}(N_t = n), \forall t \geq 0, \forall n \in \{0, 1, \cdots\}$. Note that

$$p_0(t+h) = \mathbf{P}(N_{t+h} = 0) = \mathbf{P}(N_t = 0, N_{t+h} - N_t = 0)$$
$$\stackrel{(a)}{=} \mathbf{P}(N_t = 0)\mathbf{P}(N_{t+h} - N_t = 0) \stackrel{(b)}{=} p_0(t)\mathbf{P}(N_h = 0)$$

where (a) and (b) follow since $\{N_t\}_{t\geq 0}$ has independent and stationary increments.

Poisson Process

Proof of Proposition 21.1 (Continued)

- Since $\mathbf{P}(N_h = 0) = 1 - \mathbf{P}(N_h = 1) - \mathbf{P}(N_h \geq 2) = 1 - \lambda h + o(h)$, we get:

$$p_0(t+h) = p_0(t)(1 - \lambda h) + o(h)$$

- Taking the limit $h \to 0$, we get:

$$p_0'(t) = \lim_{h \to 0} \frac{p_0(t+h) - p_0(t)}{h} = -\lambda p_0(t)$$

- The general solution is $p_0(t) = C_0 e^{-\lambda t}, \forall t \geq 0$. Since $p_0(0) = \mathbf{P}(N_0 = 0) = 1$, we have $C_0 = 1$. Thus, $p_0(t) = e^{-\lambda t}, \forall t \geq 0$.

Poisson Process

Proof of Proposition 21.1 (Continued)

- Similarly, if $n \geq 1$, then we have the following.

$$
\begin{aligned}
p_n(t+h) &= \mathbf{P}(N_t = n, N_{t+h} - N_t = 0) \\
&\quad + \mathbf{P}(N_t = n-1, N_{t+h} - N_t = 1) + \mathbf{P}(N_{t+h} = n, N_{t+h} - N_t \geq 2)
\end{aligned}
$$

- Using the fact that $\{N_t\}_{t \geq 0}$ has independent and stationary increments, and $\mathbf{P}(N_h = 0) = 1 - \lambda h + o(h)$, and $\mathbf{P}(N_h = 1) = \lambda h + o(h)$, we obtain

$$
\begin{aligned}
\mathbf{P}(N_t = n, N_{t+h} - N_t = 0) &= p_n(t)(1 - \lambda h) + o(h) \\
\mathbf{P}(N_t = n-1, N_{t+h} - N_t = 1) &= \lambda h p_{n-1}(t) + o(h)
\end{aligned}
$$

Poisson Process

Proof of Proposition 21.1 (Continued)

- Finally, $\mathbf{P}(N_{t+h} = n, N_{t+h} - N_t \geq 2) = o(h)$. Combining, we get:

$$p_n(t+h) = p_n(t)(1 - \lambda h) + \lambda h p_{n-1}(t) + o(h)$$

- Taking the limit $h \to 0$, we obtain:

$$p_n'(t) = \lim_{h \to 0} \frac{p_n(t+h) - p_n(t)}{h} = -\lambda p_n(t) + \lambda p_{n-1}(t)$$

- Define $H_n : p_n(t) = e^{-\lambda t}(\lambda t)^n / n!, \forall t \geq 0$. Clearly, H_0 is true.

Poisson Process

Proof of Proposition 21.1 (Continued)

- Assuming H_n to be true, we can write the differential equation for $p_{n+1}(t)$ as follows.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}p_{n+1}(t)\right) = e^{\lambda t}\left\{p'_{n+1}(t) + \lambda p_{n+1}(t)\right\} = e^{\lambda t}\lambda p_n(t) = \frac{\lambda^{n+1}t^n}{n!}$$

- The general solution is $p_{n+1}(t) = e^{-\lambda t}\left(C_{n+1} + (\lambda t)^{n+1}/(n+1)!\right)$.
- Since $p_{n+1}(0) = 0$, $\forall n \in \{0, 1, \cdots\}$, we must have $C_{n+1} = 0$.
- Thus, $p_{n+1}(t) = e^{-\lambda t}(\lambda t)^{n+1}/(n+1)!$, $\forall t \geq 0$, i.e., H_{n+1} is true.
- Via induction, H_n is true $\forall n \in \{0, 1, \cdots\}$.

Poisson Process

Proposition 21.2

Consider a Poisson process $\{N_t\}_{t\geq 0}$ with rate λ. Let S_n be the time of occurrence of the nth event. Moreover, let $X_1 = S_1$, and $X_n = S_n - S_{n-1}$ be the inter-arrival time between the nth and the $(n-1)$th events. Show that

(a) X_n's are independent exponential random variables with parameter λ.

(b) $S_n \sim \text{Gamma}(n, \lambda)$.

Proof of Proposition 21.2

- Assume that the underlying probability space is $(\Omega, \mathcal{F}, \mathbf{P})$. Note that $\mathbf{P}(X_1 > t) = \mathbf{P}(N_t = 0) = e^{-\lambda t}$. Thus, $X_1 \sim \exp(\lambda)$.

Poisson Process

Proof of Proposition 21.2 (Continued)

- Moreover, the following holds $\forall s, t \geq 0$.

$$\mathbf{P}(X_2 > t \mid X_1 = s) = \mathbf{P}(N_{t+s} - N_s = 0 \mid X_1 = s) \overset{(a)}{=} \mathbf{P}(N_{t+s} - N_s = 0) = e^{-\lambda t}$$

  where (a) follows because of independent increments. Hence, $X_2 \sim \exp(\lambda)$, and X_2 is independent of X_1. Repeating the process, we conclude that X_n's are independent and $X_n \sim \exp(\lambda)$, $\forall n \in \mathbb{N}$.
- Since $S_n = X_1 + \cdots + X_n$, we conclude that $S_n \sim \text{Gamma}(n, \lambda)$.

Poisson Process

Proof of Proposition 21.2 (Continued)

- Alternatively, note that $F_{S_n}(t) = \mathbf{P}(S_n \leq t) = \mathbf{P}(N_t \geq n)$. Thus,

$$f_{S_n}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=n}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\right)$$
$$= \sum_{k=n}^{\infty} e^{-\lambda t}\frac{\lambda^k t^{k-1}}{(k-1)!} - \sum_{k=n}^{\infty} e^{-\lambda t}\frac{\lambda^{k+1} t^k}{k!} = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$

Stationarity

Definition

Consider a random process $X = \{X_t\}_{t \in \mathbb{T}}$, where either $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$. The process X is said to be stationary if $\forall n \in \mathbb{N}$, and $\forall s, t_1, \cdots, t_n \in \mathbb{T}$, the joint CDFs of $(X_{t_1}, \cdots, X_{t_n})$ and $(X_{t_1+s}, \cdots, X_{t_n+s})$ are the same.

- In simple terms, a process X is stationary if its statistical properties are invariant under translation in time.

Corollary

Let $X = \{X_t\}_{t \in \mathbb{T}}$ be a second-order process, i.e., $\mathbf{E}[X_t^2] < \infty, \forall t \in \mathbb{T}$. If it is stationary,

- the mean $\mu_X(t) = \mathbf{E}[X_t]$ must be the same for all $t \in \mathbb{T}$.

Stationarity

Corollary

- the correlation $R_X(s, s+t)$ must only depend on t, $\forall s, t \in \mathbb{T}$ (denoted as $R_X(t)$).
- the covariance $C_X(s, s+t)$ must only depend on t, $\forall s, t \in \mathbb{T}$ (denoted as $C_X(t)$).

Definition

Consider a second-order process $X = \{X_t\}_{t \in \mathbb{T}}$, where either $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$. This process is called a wide sense stationary (WSS) process if

(a) the mean $\mu_X(t)$ is the same for all $t \in \mathbb{T}$.

(b) the correlation $R_X(s, s+t)$ only depends on t, $\forall s, t \in \mathbb{T}$.

Stationarity

Note

- A stationary process must be a WSS process, but the converse is not true in general.

Example

Consider a random process $X = \{X_t\}_{t\in\mathbb{R}}$, where $X_t = A\cos(\omega_c t + \Theta)$. The parameter ω_c is a non-zero constant, and A is a positive random variable with finite variance. Finally, Θ is a random variable that is independent of A. Find out if X is a stationary or a WSS process if

(a) $\Theta \in \text{Unif}(0, 2\pi)$

(b) Θ takes $\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ values with equal probabilities.

Stationarity

Solution to Part (a)

- Note that for any $t, s \in \mathbb{R}$, $X_{t+s} = A\cos(\omega_c t + \tilde{\Theta})$, where $\tilde{\Theta} = (\omega_c s + \Theta) \bmod 2\pi$.
- If $\Theta \sim \text{Unif}(0, 2\pi)$, then $\tilde{\Theta} \sim \text{Unif}(0, 2\pi)$.
- In this case, $(X_{t_1}, \cdots, X_{t_n})$ and $(X_{t_1+s}, \cdots, X_{t_n+s})$ must have the same joint CDFs for any $n \in \mathbb{N}$ and any $s, t_1, \cdots, t_n \in \mathbb{R}$. Therefore, X must be stationary.

Solution to Part (b)

- Assume the underlying probability space is $(\Omega, \mathcal{F}, \mathbf{P})$.
- Note that $\mathbf{P}(X_0 = 0) = \mathbf{P}\left(\Theta \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}\right) = 0.5$.

Stationarity

Solution to Part (b)

- Choose t such that $\omega_c t = \frac{\pi}{4}$. In this case, $\mathbf{P}(X_t = 0) = 0$. Thus, X is not stationary.
- The mean function of X can be computed as:

$$\begin{aligned}\mu_X(t) &= \mathbf{E}[A]\mathbf{E}[\cos(\omega_c t + \Theta)] \\ &= \mathbf{E}[A]\left\{\cos(\omega_c t)\mathbf{E}[\cos(\Theta)] - \sin(\omega_c t)\mathbf{E}[\sin(\Theta)]\right\}\end{aligned}$$

- Note that $\mathbf{E}[\cos(\Theta)] = 0 = \mathbf{E}[\sin(\Theta)]$. Thus $\mu_x(t) = 0, \forall t \in \mathbb{R}$.
- Moreover, $\mathbf{E}[\cos(2\Theta)] = 0 = \mathbf{E}[\sin(2\Theta)]$. This leads to the following.

Stationarity

Solution to Part (b)

$$R_X(s,t) = \mathbf{E}[A^2]\mathbf{E}[\cos(\omega_c s + \Theta)\cos(\omega_c t + \Theta)]$$
$$= \frac{1}{2}\mathbf{E}[A^2]\left\{\mathbf{E}[\cos(\omega_c(s+t) + 2\Theta)] + \cos(\omega_c(t-s))\right\}$$
$$= \frac{1}{2}\mathbf{E}[A^2]\left\{\cos(\omega_c(s+t))\mathbf{E}[\cos(2\Theta)] - \sin(\omega_c(s+t))\mathbf{E}[\sin(2\Theta)] + \cos(\omega_c(t-s))\right\}$$
$$= \frac{1}{2}\mathbf{E}[A^2]\left\{\cos(\omega_c(t-s))\right\}$$

- Note that $\mu_X(t)$ is same $\forall t \in \mathbb{R}$ and $R_x(s,t)$ is a function of $t-s$. Hence, X is WSS.