Probability and Stochastic Processes
Lecture 3: Prelude to Probability Spaces

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Basic Definitions

Sample Space

A sample space Ω is the collection of all possible outcomes of a *random experiment*.

Examples

Ω can be either finite, countably infinite, or uncountably infinite.

(a) For a double coin toss, $\Omega = \{HH, HT, TH, TT\}$.

(b) For a single die throw, $\Omega = \{1, \cdots, 6\}$.

(c) In the context of photon detection, $\Omega = \{0, 1, \cdots\}$.

(d) In the context of wealth distribution of a country, $\Omega = [0, \infty)$.

Basic Definitions

Events

Informally, an event A is a subset of the sample space Ω. If A, B are two events, then

(a) $A \cup B$ denotes the event that either A or B occurs.

(b) $A \cap B$ denotes the event that both A and B occur.

(c) Ω and ϕ respectively denote the *certain* and *impossible* events.

Example

(a) In a die throw, $\{1, 3, 5\}$ is the event that an odd number appears.

(b) In a double coin toss, $\{HH, TT\}$ is the event that both tosses yield identical outcomes.

Algebra

Motivation

- The set of all subsets of Ω is 2^Ω.
- However, all subsets of Ω may not be interesting events.
- For example: consider a die-throwing game. If an odd number appears, you lose; if an even number appears, you win.
  - Events $\{1, 3, 5\}$, $\{2, 4, 6\}$ are relevant for deciding the outcome of the game.
  - Events $\{1, 3, 4\}$, $\{1, 4\}$, etc. are not relevant.
- We focus on $\mathcal{F} \subset 2^\Omega$, as a set of meaningful events.
- Which subsets of Ω should we include in the collection $\mathcal{F}$?

Algebra

Motivation

- The *certain event* Ω must be included.
- If $A \in \mathcal{F}$, then we should include A^c (the event that A does not occur) in $\mathcal{F}$.
- If $A, B \in \mathcal{F}$, then we should include $A \cup B$ (the event that either A or B occurs) in $\mathcal{F}$.

Definition

A set $\mathcal{F} \subset 2^{\Omega}$ is called an algebra of Ω if the following conditions holds.

(a) $\Omega \in \mathcal{F}$, i.e., $\mathcal{F}$ contains Ω as one of its elements.

(b) $A \in \mathcal{F} \implies A^c \in \mathcal{F}$, i.e., $\mathcal{F}$ is closed under complement.

(c) $A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}$, i.e., $\mathcal{F}$ is closed under union.

Properties of an Algebra

Proposition 3.1

Let $\mathcal{F}$ be an algebra of Ω. The following propositions hold.

(a) $\phi \in \mathcal{F}$, i.e., $\mathcal{F}$ contains the empty set.

(b) If $A_1, \cdots, A_n \in \mathcal{F}$, then $\cup_{i=1}^n A_i \in \mathcal{F}$, i.e., $\mathcal{F}$ is closed under finite union.

(c) If $A_1, \cdots, A_n \in \mathcal{F}$, then $\cap_{i=1}^n A_i \in \mathcal{F}$, i.e., $\mathcal{F}$ is closed under finite intersection.

Proof of Proposition 3.1

(a) Since $\mathcal{F}$ is closed under complement, $\phi = \Omega^c \in \mathcal{F}$.

(b) Note that $\cup_{i=1}^n A_i = (\cup_{i=1}^{n-1} A_i) \cup A_n$. Use induction to complete the proof.

(c) Using De Morgan's law, we get $\cap_{i=1}^n A_i = (\cup_{i=1}^n A_i^c)^c$.

Example of an Algebra

Proposition 3.2

If $\mathcal{F} = \{A \subset \mathbb{N} \mid A \text{ is finite or } A^c \text{ is finite}\}$, then $\mathcal{F}$ is an algebra of $\mathbb{N}$.

Proof of Proposition 3.2

- Clearly, $\mathbb{N} \in \mathcal{F}$ as $\mathbb{N}^c = \phi$ is finite.
- If $A \in \mathcal{F}$, then either A is finite or A^c is finite. In either of these cases, $A^c \in \mathcal{F}$.
- If $A, B \in \mathcal{F}$, then two cases might arise.
  - If both A, B are finite, then $A \cup B$ is finite.
  - If at least one of A^c and B^c is finite, then $A^c \cap B^c = (A \cup B)^c$ is finite.
  - In either of the above cases, $A \cup B \in \mathcal{F}$.

σ-algebra

Motivation

- Suppose that a prisoner tries to escape the prison each day. Let the event that he escapes the prison on the nth day be A_n.
- Clearly, all A_n's are interesting events to the prisoner.
- The event that he eventually escapes is $\cup_{n=1}^{\infty} A_n$.
- If the collection of interesting events, $\mathcal{F}$, is an algebra, then the inclusion of A_n's in $\mathcal{F}$ does not guarantee the inclusion of $\cup_{n=1}^{\infty} A_n$ in $\mathcal{F}$.
- The structure of an algebra may not be rich enough to include all interesting events.

σ-algebra

Definition

A set $\mathcal{F} \subset 2^{\Omega}$ is called a σ-algebra of Ω if the following conditions hold.

(a) $\Omega \in \mathcal{F}$, i.e., $\mathcal{F}$ contains Ω as one of its elements.

(b) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, i.e., $\mathcal{F}$ is closed under complement.

(c) If $A_n \in \mathcal{F}, \forall n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$, i.e., $\mathcal{F}$ is closed under *countable* union.

Notes

- Each $A \in \mathcal{F}$ is called an $\mathcal{F}$-measurable set. Each $\mathcal{F}$-measurable set is called an event (unlike the informal definition, where each $A \subset \Omega$ is called an event).
- The tuple $(\Omega, \mathcal{F})$ is called a measurable space.

σ-algebra

Note

- Every σ-algebra of Ω is an algebra of Ω, but not conversely.

Proposition 3.3

Let $\mathcal{F} = \{A \subset \mathbb{N} \mid A \text{ is finite or } A^c \text{ is finite}\}$. Prove that $\mathcal{F}$ is not a σ-algebra of $\mathbb{N}$.

Proof of Proposition 3.3

- Note that $2\mathbb{N} \notin \mathcal{F}$ since neither $2\mathbb{N}$ nor $\mathbb{N} \setminus 2\mathbb{N}$ is finite.
- Moreover, $2\mathbb{N} = \cup_{n \in \mathbb{N}} \{2n\}$, and $\{2n\} \in \mathcal{F}$, $\forall n \in \mathbb{N}$.
- Therefore, $\mathcal{F}$ is not closed under countable union.

σ-algebra

Examples

- If the sample space is $\Omega = \{a, b, c\}$, then
  - (a) $\mathcal{F} = \{\phi, \Omega, \{a, b\}, \{c\}\}$ is a σ-algebra of Ω.
  - (b) $\mathcal{F} = \{\phi, \Omega, \{a\}, \{b\}, \{c\}, \{a, b\}\}$ is not a σ-algebra of Ω.
- If the sample space is $\Omega = [0, 1]$, then
  - (a) $\mathcal{F} = \{\phi, \Omega, [0, 0.5), [0.5, 1]\}$ is a σ-algebra of Ω.
  - (b) $\mathcal{F} = \{\phi, \Omega, [0, 0.5], [0.5, 1]\}$ is not a σ-algebra of Ω.
- The smallest σ-algebra of Ω is $\{\phi, \Omega\}$.
- The largest σ-algebra of Ω is 2^{Ω}.

σ-algebra

Proposition 3.4

If $\mathcal{F}$ is a σ-algebra of Ω, and $A_n \in \mathcal{F}$, $\forall n \in \mathbb{N}$, then $\cap_{n\in\mathbb{N}} A_n \in \mathcal{F}$. In other words, $\mathcal{F}$ is closed under countable intersection.

Proof of Proposition 3.4

- Note that $A_n^c \in \mathcal{F}$, $\forall n \in \mathbb{N}$ since $\mathcal{F}$ is closed under complement.
- Moreover, $\cup_{n\in\mathbb{N}} A_n^c \in \mathcal{F}$ since $\mathcal{F}$ is closed under countable union.
- Finally, $\cap_{n\in\mathbb{N}} A_n = (\cup_{n\in\mathbb{N}} A_n^c)^c$ due to De Morgan's law, and $(\cup_{n\in\mathbb{N}} A_n^c)^c \in \mathcal{F}$ since $\mathcal{F}$ is closed under complement.

σ-algebra

Proposition 3.5

Let $\{\mathcal{F}_\alpha\}_{\alpha\in I}$ be an arbitrary collection of σ-algebras of Ω. Their intersection $\cap_{\alpha\in I}\mathcal{F}_\alpha$ is also a σ-algebra of Ω.

Proof of Proposition 3.5

- Note that $\Omega \in \mathcal{F}_\alpha, \forall\alpha \in I$ implies that $\Omega \in \cap_{\alpha\in I}\mathcal{F}_\alpha$.
- $A \in \cap_{\alpha\in I}\mathcal{F}_\alpha \implies A \in \mathcal{F}_\alpha, \forall\alpha \in I \implies A^c \in \mathcal{F}_\alpha, \forall\alpha \in I \implies A^c \in \cap_{\alpha\in I}\mathcal{F}_\alpha$. Finally,

$$\begin{aligned} A_n \in \cap_{\alpha\in I}\mathcal{F}_\alpha, \forall n \in \mathbb{N} &\implies A_n \in \mathcal{F}_\alpha, \forall n \in \mathbb{N}, \forall\alpha \in I \\ &\implies \cup_{n\in\mathbb{N}} A_n \in \mathcal{F}_\alpha, \forall\alpha \in I \implies \cup_{n\in\mathbb{N}} A_n \in \cap_{\alpha\in I}\mathcal{F}_\alpha \end{aligned}$$

σ-algebra

Note

- If $\mathcal{F}_1, \mathcal{F}_2$ are σ-algebras of Ω, then $\mathcal{F}_1 \cup \mathcal{F}_2$ may not be a σ-algebra of Ω.

Example

Let $\Omega = \{a, b, c\}$, $\mathcal{F}_1 = \{\phi, \Omega, \{a\}, \{b, c\}\}$, and $\mathcal{F}_2 = \{\phi, \Omega, \{a, b\}, \{c\}\}$.

(a) $\mathcal{F}_1, \mathcal{F}_2$ are σ-algebras of Ω.

(b) $\mathcal{F}_1 \cap \mathcal{F}_2 = \{\phi, \Omega\}$ is a σ-algebra of Ω.

(c) $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\phi, \Omega, \{a\}, \{b, c\}, \{a, b\}, \{c\}\}$ is not a σ-algebra of Ω.

Generated σ-algebra

Definition

Let $\mathcal{A} \subset 2^{\Omega}$ and $\{\mathcal{F}_{\alpha}\}_{\alpha \in I}$ be the collection of all σ-algebras of Ω that contain $\mathcal{A}$ i.e., $\mathcal{A} \subset \mathcal{F}_{\alpha}, \forall \alpha \in I$. The smallest σ-algebra containing $\mathcal{A}$, denoted as $\sigma(\mathcal{A})$, is defined as:

$$\sigma(\mathcal{A}) = \bigcap_{\alpha \in I} \mathcal{F}_{\alpha}$$

Notes

- $\sigma(\mathcal{A})$ is also known as the σ-algebra generated by $\mathcal{A}$.
- It is guaranteed that $I \neq \phi$ since 2^{Ω} contains $\mathcal{A}$.

Generated σ-algebra

Examples

(a) If $\Omega = \{a, b, c\}$ and $\mathcal{A} = \{\{a\}, \Omega\}$, then $\sigma(\mathcal{A}) = \{\phi, \Omega, \{a\}, \{b, c\}\}$.

(b) If $\Omega = [0, 1]$ and $\mathcal{A} = \{[0, 0.4), \{0.4\}\}$, then

$$\sigma(\mathcal{A}) = \{\phi, \Omega, [0, 0.4), [0.4, 1], \{0.4\}, [0, 0.4], (0.4, 1], [0, 0.4) \cup (0.4, 1]\}$$

Notes

- If $\mathcal{A}$ is a σ-algebra of Ω, then $\sigma(\mathcal{A}) = \mathcal{A}$.
- In particular, $\sigma(\sigma(\mathcal{A})) = \sigma(\mathcal{A})$ for any $\mathcal{A} \subset 2^{\Omega}$.

Generated σ-algebra

Proposition 3.6

If $\mathcal{A} \subset \mathcal{B} \subset 2^{\Omega}$, then $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$.

Proof of Proposition 3.6

Let $\{\mathcal{F}_\alpha\}_{\alpha \in I}$ and $\{\mathcal{F}_\alpha\}_{\alpha \in J}$ be the collections of σ-algebras containing $\mathcal{A}$ and $\mathcal{B}$ respectively. Since $\mathcal{A} \subset \mathcal{B}$, we must have $J \subset I$. Hence,

$$\sigma(\mathcal{A}) = \bigcap_{\alpha \in I} \mathcal{F}_\alpha \subset \bigcap_{\alpha \in J} \mathcal{F}_\alpha = \sigma(\mathcal{B})$$

Borel σ-algebra

Definition

Borel σ-algebra of $\mathbb{R}$, denoted as $\mathcal{B}(\mathbb{R})$, is the σ-algebra generated by the collection of all open semi-infinite intervals of $\mathbb{R}$ of the form $(-\infty, a)$, where $a \in \mathbb{R}$.

- If $A \in \mathcal{B}(\mathbb{R})$, then A is called a Borel (measurable) set.

Proposition 3.7

$\mathcal{B}(\mathbb{R})$ contains semi-infinite intervals of the following forms, where $a \in \mathbb{R}$.

(a) $[a, \infty)$

(b) (a, ∞), $(-\infty, a]$

Borel σ-algebra

Proof of Proposition 3.7(a)

Note that $[a, \infty)$ is the complement of $(-\infty, a)$.

Proof of Proposition 3.7(b)

(a) Recall that any σ-algebra is closed under countable unions. Moreover,

$$(a, \infty) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, \infty\right)$$

(b) $(-\infty, a]$ is the complement of (a, ∞).

Borel σ-algebra

Proposition 3.8

$\mathcal{B}(\mathbb{R})$ contains intervals of the following forms, where $a < b$, and $a, b \in \mathbb{R}$.

$$(a, b), \ (a, b], \ [a, b), \ [a, b]$$

Proof of Proposition 3.8

(a) $(a, b) = (-\infty, b) \cap (a, \infty)$

(b) $(a, b] = (-\infty, b] \cap (a, \infty)$

(c) $[a, b) = (-\infty, b) \cap [a, \infty)$

(d) $[a, b] = (-\infty, b] \cap [a, \infty)$

Borel σ-algebra

Proposition 3.9

(a) $\mathcal{B}(\mathbb{R})$ contains all singleton sets of the form $\{a\}$, where $a \in \mathbb{R}$.

(b) $\mathcal{B}(\mathbb{R})$ contains all countable subsets of $\mathbb{R}$.

Proof of Proposition 3.9(a)

- Recall that $(-\infty, a] \in \mathcal{B}(\mathbb{R})$ and $[a, \infty) \in \mathcal{B}(\mathbb{R})$.
- Therefore, $\{a\} = (-\infty, a] \cap [a, \infty) \in \mathcal{B}(\mathbb{R})$.

Proof of Proposition 3.9(b)

Countable subsets of $\mathbb{R}$ are countable unions of singleton sets.

Borel σ-algebra

Proposition 3.10

$\mathcal{B}(\mathbb{R})$ can equivalently be defined as the σ-algebra generated by the collection of all intervals of $\mathbb{R}$ of the form $(-\infty, a]$, where $a \in \mathbb{R}$.

Proof of Proposition 3.10

- Let $\mathcal{G} = \{(-\infty, a] | a \in \mathbb{R}\}$ and $\mathcal{H} = \{(-\infty, a) | a \in \mathbb{R}\}$.
- The definition of $\mathcal{B}(\mathbb{R})$ states that $\sigma(\mathcal{H}) = \mathcal{B}(\mathbb{R})$.
- Proposition 3.7(b) shows that $\mathcal{G} \subset \sigma(\mathcal{H})$.
- Moreover, $\mathcal{H} \subset \sigma(\mathcal{G})$ since $(-\infty, a) = \cup_{n=1}^{\infty}(-\infty, a - \frac{1}{n}]$.
- Applying Proposition 3.6, we obtain $\sigma(\mathcal{G}) \subset \sigma(\mathcal{H})$ and $\sigma(\mathcal{H}) \subset \sigma(\mathcal{G})$.

Borel σ-algebra

Notes

- Similar to $\mathcal{B}(\mathbb{R})$, one can define $\mathcal{B}([0,1])$ as the σ-algebra generated by the collection of all intervals of the form $(-\infty, a) \cap [0,1]$, where $a \in \mathbb{R}$.
- One can show that $\mathcal{B}(\mathbb{R}) \neq 2^{\mathbb{R}}$.
- Proposition 3.9(b) shows that if $A \notin \mathcal{B}(\mathbb{R})$, then A must be uncountable.
- However, $\mathcal{B}(\mathbb{R})$ may contain some uncountable sets, e.g., $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{B}(\mathbb{R})$.
- Constructing $A \notin \mathcal{B}(\mathbb{R})$ is not easy. One famous example is Vitali sets.