Probability and Stochastic Processes
Lecture 5: A Primer on Combinatorial Probability

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Fundamental of Combinatorics

Counting Basics

- The number of ways n *distinct* objects can be ordered is $n! = n \times (n-1) \times \cdots \times 1$.
- The number of ways n *distinct* objects can be divided into k *distinct* groups such that the ith group contains n_i objects and $\sum_{i=1}^{k} n_i = n$, is

$$\frac{n!}{n_1!n_2!\cdots n_k!} = \binom{n}{n_1, n_2, \cdots, n_k}$$

- The above term is called a multinomial coefficient because of its appearance in the expansion of $(x_1 + \cdots + x_k)^n$.

Fundamentals of Combinatorics

Choosing r Out of n Distinct Objects

- The task is the same as dividing n distinct objects into two groups A and B containing r and $n-r$ objects, respectively. The group A objects are chosen, whereas group B objects are discarded. The number of ways to accomplish this goal is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$$

- The above term is called a binomial coefficient.
- If $r < 0$ or $r > n$, then $\binom{n}{r} = 0$. Similar rules apply to multinomial coefficients.

Proposition 5.1

The following holds for positive integers $n, n_1, \cdots, n_k$ such that $n = n_1 + \cdots + n_k$.

(a) $$\binom{n}{n_1, n_2, \cdots, n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_k}{n_k}$$

(b) $$\binom{n}{n_1, n_2, \cdots, n_k} = \binom{n-1}{n_1-1, n_2, \cdots, n_k} + \cdots + \binom{n-1}{n_1, n_2, \cdots, n_k-1}$$

Fundamentals of Combinatorics

Proof of Proposition 5.1(a)

- Consider the task of dividing n distinct objects into k distinct groups, where the ith group contains n_i objects, and $\sum_{i=1}^{k} n_i = n$.
- Group 1's elements can be chosen in $\binom{n}{n_1}$ ways, group 2's in $\binom{n-n_1}{n_2}$ ways, etc.

Proof of Proposition 5.1(b)

- Pick an object O_1. If it belongs to the ith group, then the remaining $n-1$ distinct objects need to be divided into k distinct groups such that the ith group contains $n_i - 1$ elements, and the jth group contains n_j elements, where $j \neq i$.

Fundamentals of Combinatorics

Example

Find the number of arrangements of the letters EQUIPPED.

Solution

- Had all letters been distinct, the number of arrangements would have been $8!$. However, two Es and two Ps are identical. Hence, the required number is $8!/(2!2!)$.

Fundamentals of Combinatorics

Example

Find the number of positive integer solutions of the equation $x_1 + \cdots + x_r = n$, where n, r are positive integers.

Solution

- We need to divide n *identical* objects into r non-empty groups. Assume that n objects are lined up in a row. We need to put $r-1$ barriers into $n-1$ spaces between adjacent objects. The number is $\binom{n-1}{r-1}$.

Fundamentals of Combinatorics

Example

Ten people are to be seated in a row. How many arrangements are possible if

(a) two specific persons A and B must sit together?

(b) there are 5 men and 5 women, and no two persons of the same gender sit together?

(c) there are 5 married couples, and each couple must sit together?

Solution

(a) Treat $A + B$ as a unit. The remaining 8 people and $A + B$ unit can be arranged in $9!$ ways. The unit $A + B$ can be arranged in $2!$ ways. The number is $9! \times 2!$.

(b) Either men occupy odd positions and women even positions, or vice versa.

- In each case, 5 men can be arranged in 5 positions in $5!$ ways. Similarly, 5 women can be arranged in 5 positions in $5!$ ways. The number is $2 \times 5! \times 5!$.

(c) The 5 couples can be arranged in $5!$ ways. Moreover, each couple can be arranged in $2!$ ways. The total number is $5! \times (2!)^5$.

Fundamentals of Combinatorics

Example

A professor wants to divide a group of 40 students into 10 groups, each containing 4 students. Find the number of ways this can be accomplished.

Solution

- Note that 40 students can be divided into 10 *distinct* groups, each containing 4 students in $40!/(4!)^{10}$ ways.
- Once the students are assigned to groups, all $10!$ arrangements of the groups are considered identical. Hence, the required number is $40!/((4!)^{10}10!)$.

Combinatorial Probability

Basic Principle

- Consider a probability space $(\Omega, 2^{\Omega}, \mathbf{P})$, where Ω is finite, i.e., $|\Omega| < \infty$, and all outcomes are equally likely, i.e.,

$$\mathbf{P}(\{\omega\}) = \frac{1}{|\Omega|}, \ \forall \omega \in \Omega$$

- If $A \in 2^{\Omega}$, then $\mathbf{P}(A) = |A|/|\Omega|$. If $\omega \in A$, then ω is called a favorable outcome. Therefore, $\mathbf{P}(A)$, in this case, is the ratio of number of favorable outcomes and total number of outcomes.

Combinatorial Probability

Example

If there are 10 people in a room, what is the probability that none of their birthdays coincide? Assume all outcomes are equally likely.

Solution

- Total number of outcomes is $(365)^{10}$.
- There are 365 possibilities for 1st person's birthday, 364 possibilities for 2nd person's, and so on. Therefore, the number of favorable outcomes is $365 \times 364 \times \cdots \times 356$.
- The probability is $(365 \times \cdots \times 356)/(365)^{10}$.

Combinatorial Probability

Example

- An urn contains 4 red and 6 white balls. Two balls are chosen without replacement. If all outcomes are equally likely, what is the probability that
  (a) one of the chosen balls is red, and the other one is white?
  (b) both balls are red?
- If the balls are chosen with replacement, find the probability that
  (c) the first chosen ball is red and the second one is white.
  (d) one of the chosen balls is red, and the other one is white.
  (e) both balls are red.

Combinatorial Probability

Solution

- For (a) and (b), the number of (unordered) outcomes is $\binom{10}{2} = 45$.
  - (a) Required probability is $\binom{4}{1}\binom{6}{1}/\binom{10}{2} = 24/45$.
  - (b) Required probability is $\binom{4}{2}/\binom{10}{2} = 6/45$.
- For (c), (d), and (e), the number of (ordered) outcomes is 10^2.
  - (c) Required probability is $4 \times 6/10^2 = 0.24$.
  - (d) Required probability is $2 \times 4 \times 6/10^2 = 0.48$.
  - (e) Required probability is $4 \times 4/10^2 = 0.16$.

Combinatorial Probability

Example

In a football tournament, 4 teams from group A and 4 teams from group B are arranged to play 4 matches. If all pairings are equally likely, calculate the probability that there are no intra-group pairings.

Solution

- Let $\{A_1, A_2, A_3, A_4\}$ be group A teams, and $\{B_1, B_2, B_3, B_4\}$ be group B teams.
- Note that A_1 can be paired with any of the 4 group B teams. Given this pairing, A_2 can be paired with any of the remaining 3 group B teams, and so on.
- Number of favorable outcomes is $4 \times 3 \times 2 \times 1 = 24$.

Combinatorial Probability

Solution (Continued)

- To obtain the total number of outcomes, note that the first pair can be selected in $\binom{8}{2}$ ways, the second one in $\binom{6}{2}$ ways, and so on. Moreover, once the pairs are formed, their $4!$ arrangements are treated as identical. The total number of outcomes is

$$\frac{\binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2}}{4!} = 105$$

- The required probability is $24/105$.

Combinatorial Probability

Example

Suppose that 20 men and 20 women are to be segregated into 20 pairs. If all pairings are equally likely to occur, obtain the probability that 16 of the pairs consist of people of the same gender, and the remaining 4 pairs have people of opposite genders.

Solution

- 40 people can be divided into 20 equal-sized distinct groups in $40!/(2!)^{20}$ ways. Since all 20! arrangements of the groups are treated to be the same, the total number of outcomes turns out to be
$$\frac{40!}{(2!)^{20}20!}$$

Combinatorial Probability

Solution (Continued)

- Note that 16 men can be chosen in $\binom{20}{16}$ ways. These chosen men can be divided into 8 equal-sized exchangeable groups in $16!/((2!)^8 8!)$ ways.
- Similar calculation holds for same-gendered women's groups.
- The remaining 4 men and 4 women can form mixed-gendered groups in $4!$ ways (M_1 can pair with any of the 4 women, M_2 with any of the remaining 3 women, and so on).
- The desired probability is expressed as P/Q where

$$P = \left\{ \binom{20}{16} \times \frac{16!}{(2!)^8 8!} \right\}^2 \times 4!, \quad \text{and} \quad Q = \frac{40!}{(2!)^{20} 20!}$$

Combinatorial Probability

Example

Suppose that n balls numbered 1 to n are to be put into n boxes numbered 1 to n (one ball in each box). If all permutations are equally likely, find the probability that no same-numbered balls and boxes are paired up.

Solution

- Let E_i be the event that the box i contains the ball numbered i. If $i_1, \cdots, i_k$ are distinct integers in $\{1, \cdots, n\}$, and $\mathbf{P}$ is the underlying probability measure, then

$$\mathbf{P}(E_{i_1} \cap \cdots \cap E_{i_k}) = \frac{(n-k)!}{n!}$$

Combinatorial Probability

Solution (Continued)

- Our target event is $E_1^c \cap \cdots \cap E_n^c$. Using the inclusion-exclusion principle, we get

$$\begin{aligned}
\mathbf{P}(E_1^c \cap \cdots \cap E_n^c) &= 1 - \mathbf{P}(E_1 \cup \cdots \cup E_n) \\
&= 1 - \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbf{P}(E_{i_1} \cap \cdots \cap E_{i_k}) \\
&= 1 - \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}
\end{aligned}$$

- The above expression is known as the *derangement probability* of n objects.

Combinatorial Probability

Example

Ten married couples are to be seated at a round table. If all arrangements are equally likely to occur, find the probability that no husband sits next to his wife.

Solution

- Let E_i be the event that the ith couple sits together.
- Consider the event $E_{i_1} \cap \cdots \cap E_{i_k}$ where $i_1, \cdots, i_k$ are distinct integers in $\{1, \cdots, 10\}$.
- Note that 20 people can be seated at a round table in $19!$ ways (one person can act as a pivot, and the remaining 19 persons can be seated in $19!$ ways).

Combinatorial Probability

Solution (Continued)

- To find the number of favorable outcome of $E_{i_1} \cap \cdots \cap E_{i_k}$, note that each of the k couples that sit together can be treated as single units.
- These k units, and rest $20 - 2k$ persons can be arranged circularly in $(19 - k)!$ ways.
- Moreover, each of the k couples can be arranged in $2!$ ways. Therefore, if the underlying probability measure is $\mathbf{P}$, then

$$\mathbf{P}(E_{i_1} \cap \cdots \cap E_{i_k}) = \frac{2^k(19 - k)!}{19!}$$

Combinatorial Probability

Solution (Continued)

- Our target event is $E_1^c \cap \cdots \cap E_{10}^c$. Using the inclusion-exclusion principle, we get

$$
\begin{aligned}
\mathbf{P}(E_1^c \cap \cdots \cap E_{10}^c) &= 1 - \mathbf{P}(E_1 \cup \cdots \cup E_{10}) \\
&= 1 - \sum_{k=1}^{10} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le 10} \mathbf{P}(E_{i_1} \cap \cdots \cap E_{i_k}) \\
&= \sum_{k=0}^{10} (-1)^k \binom{10}{k} \frac{2^k (19-k)!}{19!}
\end{aligned}
$$