Probability and Stochastic Processes
Lecture 7: Random Variables

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Motivation

Examples

(1) It is impossible for a doctor to know the exact physiological state of a patient (this requires knowledge of every biochemical process inside each cell). However, some information about her health can be obtained by examining numerical indicators, such as heart rate and blood pressure.

(2) It is impossible to describe the states of every molecule present in the air. However, some information about the local weather can be obtained by examining numeric indicators such as pressure, temperature, and humidity.

Motivation

Towards the Definition of Random Variables

- In each of the previous examples, there is an underlying measurable space $(\Omega, \mathcal{F})$, which is not directly observable.
- Each realized outcome $\omega \in \Omega$ is observable via a function of the form $X : \Omega \to \mathbb{R}$. For example, a deteriorated health condition may be indicated by a high heart rate.
- Since X takes values in $\mathbb{R}$, from an observer's perspective, meaningful events must lie in some σ-algebra of $\mathbb{R}$. Typically, this is taken as $\mathcal{B}(\mathbb{R})$.
- X should be such that each event $B \in \mathcal{B}(\mathbb{R})$ translates to an event in $\mathcal{F}$ i.e.,

$$X^{-1}(B) = \{\omega \in \Omega \mid X(\omega) \in B\} \in \mathcal{F}$$

Random Variables

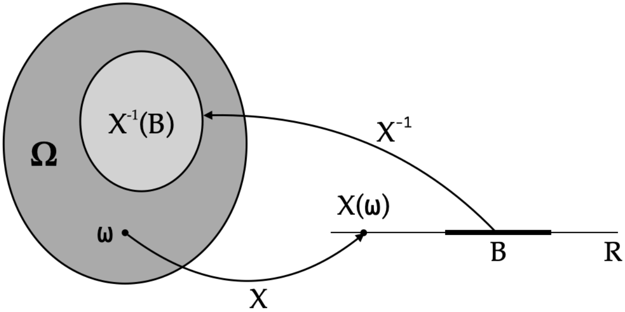

Figure: Visualization of Random Variables

Random Variables

Definition

A random variable X associated with a measurable space $(\Omega, \mathcal{F})$ is a function $X : \Omega \to \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F},\ \forall B \in \mathcal{B}(\mathbb{R})$. Such a function is also called an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function, an $\mathcal{F}$-measurable function, or simply a measurable function.

Note

- A random variable is neither random nor a variable; it is a deterministic function.

Non-Example

Let $\Omega = \{a, b, c\}$, and $\mathcal{F} = \{\phi, \Omega, \{a, b\}, \{c\}\}$. If $X : \Omega \to \mathbb{R}$ is such that $X(a) = 1$, and $X(b) = X(c) = 0$, then X is not a random variable on $(\Omega, \mathcal{F})$ since $X^{-1}(\{0\}) = \{b, c\} \notin \mathcal{F}$.

Random Variables

Example

Let $\Omega = \{a, b, c\}$, and $\mathcal{F} = \{\phi, \Omega, \{a\}, \{b, c\}\}$. If $X : \Omega \to \mathbb{R}$ is such that $X(a) = 1$, $X(b) = X(c) = 0$, then show that X is a random variable associated with $(\Omega, \mathcal{F})$.

Solution

- Choose an arbitrary $B \in \mathcal{B}(\mathbb{R})$. Note that

$$\mathcal{F} \ni X^{-1}(B) = \begin{cases} \phi & \text{if } 0, 1 \notin B \\ \{b, c\} & \text{if } 0 \in B \text{ and } 1 \notin B \\ \{a\} & \text{if } 0 \notin B \text{ and } 1 \in B \\ \Omega & \text{if } 0, 1 \in B \end{cases}$$

Random Variables

Notes

- In general, verifying that $X^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R})$ can be difficult when proving that the function $X : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable.
- The following proposition provides a simpler characterization of measurability, whose proof is left as an exercise.

Proposition 7.1

Let $(\Omega, \mathcal{F})$ be a measurable space, and $X : \Omega \to \mathbb{R}$ be such that $X^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{P}$ for some $\mathcal{P} \subset 2^{\mathbb{R}}$. If $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$, then X is a random variable associated with $(\Omega, \mathcal{F})$, i.e., X is $\mathcal{F}$-measurable.

Random Variables

Corollary 7.2

Let $(\Omega, \mathcal{F})$ be a measurable space, and $X : \Omega \to \mathbb{R}$ be arbitrary. If $X^{-1}((a, \infty)) \in \mathcal{F}$, $\forall a \in \mathbb{R}$, then X is a random variable associated with $(\Omega, \mathcal{F})$, i.e., X is $\mathcal{F}$-measurable.

Proof of Corollary 7.2

- Recall that if $\mathcal{P} = \{(a, \infty) \mid a \in \mathbb{R}\}$, then $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$.

Note

- Similar result holds for other $\mathcal{P} \subset 2^{\mathbb{R}}$ that generate $\mathcal{B}(\mathbb{R})$. For example, one can take $\mathcal{P} = \{(-\infty, a) \mid a \in \mathbb{R}\}$ or $\mathcal{P} = \{(a, b) \mid a, b \in \mathbb{R}, a < b\}$.

Random Variables

Example

Let $(\Omega, \mathcal{F})$ be a measurable space, where $\Omega = [0,1]$, and $\mathcal{F} = \mathcal{B}([0,1])$. Let $X : \Omega \to \mathbb{R}$ be such that $X(\omega) = 2\omega$, $\forall \omega \in [0,1]$. Prove that X is a random variable on $(\Omega, \mathcal{F})$.

Solution

- Directly proving that $X^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R})$ is difficult. Note that

$$X^{-1}((a,\infty)) = \{\omega \in [0,1] \mid X(\omega) > a\} = [0,1] \cap \left(\frac{a}{2}, \infty\right) \in \mathcal{F}$$

- Since the above conclusion holds $\forall a \in \mathbb{R}$, X must be $\mathcal{F}$-measurable.

Random Variables

Example

Let A be a Borel set, i.e., $A \in \mathcal{B}(\mathbb{R})$. Show that $A + 1 = \{a + 1 \mid a \in A\}$ is also a Borel set.

Solution

- Let $X : \mathbb{R} \to \mathbb{R}$ be such that $X(\omega) = \omega - 1, \forall \omega \in \mathbb{R}$. Note that for any $B \subset \mathbb{R}$,

$$X^{-1}(B) = \{\omega \in \mathbb{R} \mid \omega - 1 \in B\} = B + 1$$

- Observe that $X^{-1}((a, \infty)) = (a + 1, \infty) \in \mathcal{B}(\mathbb{R}), \forall a \in \mathbb{R}$. Therefore, X is a random variable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Hence, $X^{-1}(A) = A + 1 \in \mathcal{B}(\mathbb{R})$.

σ-algebra Generated by a Random Variable

Definition

Let $(\Omega, \mathcal{F})$ be a measurable space, and $X : \Omega \to \mathbb{R}$. The σ-algebra generated by X is:

$$\sigma(X) = \{X^{-1}(B) \mid B \in \mathcal{B}(\mathbb{R})\}$$

Clearly, X is a random variable associated with $(\Omega, \mathcal{F})$ if and only if $\sigma(X) \subset \mathcal{F}$.

Example

Let $\Omega = \{a, b, c\}$ and $X_1, X_2 : \Omega \to \mathbb{R}$ be such that $X_1(a) = 0, X_1(b) = X_1(c) = 1$, and $X_2(a) = 5$, $X_2(b) = X_2(c) = 10$. Find $\sigma(X_1)$ and $\sigma(X_2)$.

σ-algebra Generated by a Random Variable

Solution

- Let $B \in \mathcal{B}(\mathbb{R})$ be arbitrary. Note that

$$X_1^{-1}(B) = \begin{cases} \phi & \text{if } 0, 1 \notin B \\ \{a\} & \text{if } 0 \in B, 1 \notin B \\ \{b, c\} & \text{if } 0 \notin B, 1 \in B \\ \Omega & \text{if } 0, 1 \in B \end{cases} \quad \text{and} \quad X_2^{-1}(B) = \begin{cases} \phi & \text{if } 5, 10 \notin B \\ \{a\} & \text{if } 5 \in B, 10 \notin B \\ \{b, c\} & \text{if } 5 \notin B, 10 \in B \\ \Omega & \text{if } 5, 10 \in B \end{cases}$$

- We have $\sigma(X_1) = \sigma(X_2) = \{\phi, \Omega, \{a\}, \{b, c\}\}$. Thus, to obtain $\sigma(X)$, it is not important to know *what* values X takes but *how* these values are assigned.

σ-algebra Generated by a Random Variable

Proposition 7.3

Let $X : \Omega \to \mathbb{R}$ be arbitrary. The set $\sigma(X)$ is indeed a σ-algebra of Ω.

Proof of Proposition 7.3

- Since $\mathbb{R} \in \mathcal{B}(\mathbb{R})$, we have $\Omega = X^{-1}(\mathbb{R}) \in \sigma(X)$.
- If $A \in \sigma(X)$, then $A = X^{-1}(B)$ for some $B \in \mathcal{B}(\mathbb{R})$. Clearly, $B^c \in \mathcal{B}(\mathbb{R})$. Note that $A^c = [X^{-1}(B)]^c = X^{-1}(B^c) \in \sigma(X)$.
- If $A_i \in \sigma(X)$, then $A_i = X^{-1}(B_i)$ for some $B_i \in \mathcal{B}(\mathbb{R})$. For countable I, we have $\cup_{i \in I} B_i \in \mathcal{B}(\mathbb{R})$. Thus, $\cup_{i \in I} A_i = \cup_{i \in I} X^{-1}(B_i) = X^{-1}(\cup_{i \in I} B_i) \in \sigma(X)$.

Probability Measure Induced by a Random Variable

Definition

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and X be a random variable associated with the measurable space $(\Omega, \mathcal{F})$. A function $\mathbf{P}_X : \mathcal{B}(\mathbb{R}) \to [0, 1]$ is called a probability measure induced by X if $\mathbf{P}_X(B) = \mathbf{P}(X^{-1}(B)), \forall B \in \mathcal{B}(\mathbb{R})$.

Note

Since X is measurable, $X^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R})$. Thus, the function $\mathbf{P}_X(\cdot)$ is well-defined.

Proposition 7.4

The function $\mathbf{P}_X$ is indeed a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, i.e., $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbf{P}_X)$ is a probability space.

Probability Measure Induced by a Random Variable

Example

Let $\Omega = \{a, b, c\}$, $\mathcal{F} = \{\phi, \Omega, \{a\}, \{b, c\}\}$, and $\mathbf{P} : \mathcal{F} \to [0, 1]$ be such that $\mathbf{P}(\phi) = 0$, $\mathbf{P}(\Omega) = 1$, $\mathbf{P}(\{a\}) = 0.25$, $\mathbf{P}(\{b, c\}) = 0.75$. Define $X : \Omega \to \mathbb{R}$ such that $X(a) = 1$ and $X(b) = X(c) = 2$. Compute $\mathbf{P}_X(\cdot)$.

Solution

$$\mathbf{P}_X(B) = \mathbf{P}(X^{-1}(B)) = \begin{cases} \mathbf{P}(\phi) & \text{if } 1, 2 \notin B \\ \mathbf{P}(\{a\}) & \text{if } 1 \in B, 2 \notin B \\ \mathbf{P}(\{b, c\}) & \text{if } 1 \notin B, 2 \in B \\ \mathbf{P}(\Omega) & \text{if } 1, 2 \in B \end{cases} = \begin{cases} 0 & \text{if } 1, 2 \notin B \\ 0.25 & \text{if } 1 \in B, 2 \notin B \\ 0.75 & \text{if } 1 \notin B, 2 \in B \\ 1 & \text{if } 1, 2 \in B \end{cases}$$

Probability Measure Induced by a Random Variable

Proof of Proposition 7.4

- $\mathbf{P}_X(\mathbb{R}) = \mathbf{P}(X^{-1}(\mathbb{R})) = \mathbf{P}(\Omega) = 1.$
- Let $\{B_i\}_{i\in I}$ be a countable collection of disjoint $\mathcal{B}(\mathbb{R})$-measurable sets. Note that

$$\mathbf{P}_X\left(\cup_{i\in I} B_i\right) = \mathbf{P}\left(X^{-1}\left(\cup_{i\in I} B_i\right)\right) = \mathbf{P}\left(\cup_{i\in I} X^{-1}(B_i)\right)$$

- Moreover, $B_i \cap B_j = \phi \Rightarrow X^{-1}(B_i) \cap X^{-1}(B_j) = X^{-1}(B_i \cap B_j) = X^{-1}(\phi) = \phi$, i.e., $\{X^{-1}(B_i)\}_{i\in I}$ is a countable collection of disjoint $\mathcal{F}$-measurable sets. Hence,

$$\mathbf{P}_X\left(\cup_{i\in I} B_i\right) = \sum_{i\in I} \mathbf{P}(X^{-1}(B_i)) = \sum_{i\in I} \mathbf{P}_X(B_i)$$

Cumulative Distribution Function (CDF)

Motivation

- Let X be a random variable associated with the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. To specify the induced measure $\mathbf{P}_X : \mathcal{B}(\mathbb{R}) \to [0, 1]$, one needs to determine $\mathbf{P}_X(B)$, $\forall B \in \mathcal{B}(\mathbb{R})$. This is a complicated task.
- Let $\mathcal{P}$ be the collection of intervals of the form $(-\infty, a]$, where $a \in \mathbb{R}$. Recall that $\mathcal{P}$ is a π-system, and $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$.
- Using Dynkin's extension theorem, we can conclude that if $\mathbf{P}_X((-\infty, a]) = \mathbf{P}(X \leq a)$ is specified $\forall a \in \mathbb{R}$, then $\mathbf{P}_X(B)$ is uniquely determined $\forall B \in \mathcal{B}(\mathbb{R})$.

Cumulative Distribution Function (CDF)

Definition

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and X be a random variable associated with $(\Omega, \mathcal{F})$. The cumulative distribution function (CDF) of X is a function $F_X : \mathbb{R} \to [0, 1]$ such that

$$\begin{aligned} F_X(x) = \mathbf{P}_X((-\infty, x]) &= \mathbf{P}(X^{-1}(-\infty, x]) \\ &= \mathbf{P}\left(\{\omega \in \Omega \mid X(\omega) \leq x\}\right), \ \forall x \in \mathbb{R} \end{aligned}$$

where $\mathbf{P}_X$ is the induced probability measure of X.

Notations

The following notations are interchangeably used:

$$\mathbf{P}_X((-\infty, x]) = \mathbf{P}(X \in (-\infty, x]) = \mathbf{P}(X \leq x)$$

Cumulative Distribution Function (CDF)

Example

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$ and $\mathbf{P} : \mathcal{F} \to [0,1]$ is such that $\mathbf{P}([a,b]) = b^2 - a^2$, $0 \leq a \leq b \leq 1$. Let X be a random variable associated with $(\Omega, \mathcal{F})$ such that $X(\omega) = 2\omega$, $\forall \omega \in [0,1]$. Find $F_X(\cdot)$.

Solution

- Note that $F_X(x) = \mathbf{P}(X \leq x) = \mathbf{P}\left(\{\omega \in \Omega \mid X(\omega) \leq x\}\right) = \mathbf{P}(\{\omega \in \Omega \mid \omega \leq x/2\})$.

$$F_X(x) = \mathbf{P}([0,1] \cap (-\infty, x/2]) = \begin{cases} \mathbf{P}(\phi) \\ \mathbf{P}([0, x/2]) \\ \mathbf{P}([0,1]) \end{cases} = \begin{cases} 0 & \text{if } x < 0 \\ x^2/4 & \text{if } 0 \leq x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

Cumulative Distribution Function (CDF)

Example

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, where $\Omega = \{a, b, c\}$, $\mathcal{F} = 2^{\Omega}$, and $\mathbf{P} : \mathcal{F} \to [0, 1]$ is such that $\mathbf{P}(\{a\}) = \mathbf{P}(\{b\}) = 0.25$, and $\mathbf{P}(\{c\}) = 0.5$. Let X be a random variable associated with $(\Omega, \mathcal{F})$ such that $X(a) = 1$ and $X(b) = X(c) = 2$. Find $F_X(\cdot)$.

Solution

- Note that $F_X(x) = \mathbf{P}(\{\omega \in \Omega \mid X(\omega) \leq x\})$. Therefore,

$$F_X(x) = \begin{cases} \mathbf{P}(\phi) & \text{if } x < 1 \\ \mathbf{P}(\{a\}) & \text{if } 1 \leq x < 2 \\ \mathbf{P}(\Omega) & \text{if } x \geq 2 \end{cases} = \begin{cases} 0 & \text{if } x < 1 \\ 0.25 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

Cumulative Distribution Function (CDF)

Proposition 7.5

Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $F_X(\cdot)$ be its cumulative distribution function (CDF). The following properties hold.

(a) $F_X(x_1) \leq F_X(x_2)$ whenever $x_1 \leq x_2$ i.e., F_X is non-decreasing.

(b) $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$.

(c) $\lim_{h\to 0^+} F_X(x+h) = F_X(x), \forall x \in \mathbb{R}$ i.e., F_X is right-continuous.

(d) $\lim_{h\to 0^+} F_X(x-h) = F_X(x) - \mathbf{P}(X = x), \forall x \in \mathbb{R}$.

Note

F_X is both left and right continuous at x if and only if $\mathbf{P}(X = x) = 0$.

Cumulative Distribution Function (CDF)

Proof of Proposition 7.5(a)

- Let $A_1 = (-\infty, x_1]$ and $A_2 = (-\infty, x_2]$. If $x_1 \leq x_2$, then $A_1 \subset A_2$. Since $\mathbf{P}_X$ is a probability measure, we have $F_X(x_1) = \mathbf{P}_X(A_1) \leq \mathbf{P}_X(A_2) = F_X(x_2)$.

Proof of Proposition 7.5(b)

- Let $A_n = (-\infty, -n]$ and $B_n = (-\infty, n]$. Clearly, $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are decreasing and increasing sequences respectively.

$$\lim_{n\to\infty} F_X(-n) = \lim_{n\to\infty} \mathbf{P}_X(A_n) = \mathbf{P}_X(\cap_{n\in\mathbb{N}} A_n) = \mathbf{P}_X(\phi) = 0$$

$$\lim_{n\to\infty} F_X(n) = \lim_{n\to\infty} \mathbf{P}_X(B_n) = \mathbf{P}_X(\cup_{n\in\mathbb{N}} B_n) = \mathbf{P}_X(\mathbb{R}) = 1$$

Cumulative Distribution Function (CDF)

Proof of Propositions 7.5(c) and 7.5(d)

- Let $A_n = (-\infty, x + \frac{1}{n}]$ and $B_n = (-\infty, x - \frac{1}{n}]$. Clearly, $\{A_n\}_{n\in\mathbb{N}}, \{B_n\}_{n\in\mathbb{N}}$ are decreasing and increasing sequences, respectively.

$$\lim_{n\to\infty} F_X\left(x + \frac{1}{n}\right) = \lim_{n\to\infty} \mathbf{P}_X(A_n) = \mathbf{P}_X\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \mathbf{P}_X((-\infty, x]) = F_X(x)$$

$$\begin{aligned}\lim_{n\to\infty} F_X\left(x - \frac{1}{n}\right) = \lim_{n\to\infty} \mathbf{P}_X(B_n) &= \mathbf{P}_X\left(\bigcup_{n\in\mathbb{N}} B_n\right) \\ &= \mathbf{P}_X((-\infty, x)) = F_X(x) - \mathbf{P}(X = x)\end{aligned}$$

Cumulative Distribution Function (CDF)

Example

Determine why $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ defined below cannot be valid CDFs.

$$F_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}, \quad F_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x - x^2 & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

Solution

- F_1 is not right-continuous at $x = 0$. F_2 is decreasing in $(1, 2)$.

Cumulative Distribution Function (CDF)

Notes

- Sometimes a random variable is entirely characterized by its CDF, without referring to its underlying probability space.
- Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The function $\bar{F}_X(x) = 1 - F_X(x) = \mathbf{P}(X > x)$ is called the complementary CDF (CCDF) of X.
- One can show the following properties of $\bar{F}_X$.
  (a) $\bar{F}_X(x) \in [0, 1], \forall x \in \mathbb{R}$.
  (b) $\lim_{x \to -\infty} \bar{F}_X(x) = 1$, and $\lim_{x \to \infty} \bar{F}_X(x) = 0$.
  (c) $\bar{F}_X(\cdot)$ is non-increasing.
  (d) $\bar{F}_X(\cdot)$ is right-continuous.