Probability and Stochastic Processes
Lecture 8: Discrete Random Variables

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Discrete Random Variables

Definition

A random variable X associated with a measurable space $(\Omega, \mathcal{F})$ is called a discrete random variable if $\mathrm{range}(X)$ is countable.

Examples

(a) Let X be a random variable associated with $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $X(\omega) = \omega$ if $\omega \in \mathbb{N}$ and $X(\omega) = 0$ otherwise. Clearly, $\mathrm{range}(X) = \{0\} \cup \mathbb{N}$ and X is discrete.

(b) Let $\Omega = \{a, b\}$ and X be a random variable associated with $(\Omega, 2^{\Omega})$ such that $X(a) = X(b) = 1$. Clearly, X is discrete since $\mathrm{range}(X) = \{1\}$.

Probability Mass Function (PMF)

Definition

Let X be a discrete random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The probability mass function (PMF) of X is a function $p_X : \mathbb{R} \to [0,1]$ such that $\forall x \in \mathbb{R}$, $p_X(x) = \mathbf{P}(X = x) = \mathbf{P}\left(\{\omega \in \Omega \mid X(\omega) = x\}\right)$.

Notes

If X is a discrete random variable with PMF $p_X(\cdot)$ and CDF $F_X(\cdot)$, then

(a) $F_X(x) = \sum_{x_i \leq x} p_X(x_i)$ and $\sum_{x_i} p_X(x_i) = 1$ where $\sum_{x_i}$ is the sum over $\mathrm{range}(X)$.

(b) $p_X(x) = F_X(x) - \lim_{h \to 0^+} F_X(x - h)$, $\forall x \in \mathbb{R}$.

(c) $p_X(x) = 0$, $\forall x \notin \mathrm{range}(X)$ i.e., F_X is continuous at $x \notin \mathrm{range}(X)$.

Probability Mass Function (PMF)

Example

Let $(\Omega, 2^\Omega, \mathbf{P})$ be a probability space, where $\Omega = \{a, b, c, d\}$, and $\mathbf{P} : 2^\Omega \to [0, 1]$ be such that $\mathbf{P}(\{a\}) = \mathbf{P}(\{b\}) = \mathbf{P}(\{c\}) = 0.2$, and $\mathbf{P}(\{d\}) = 0.4$. Let X be an associated discrete random variable such that $X(a) = X(b) = 0$, and $X(c) = X(d) = 1$. Find p_X, F_X.

Solution

$$p_X(x) = \begin{cases} 0.4 & \text{if } x = 0 \\ 0.6 & \text{if } x = 1 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.4 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

- Note that F_X is continuous at all $x \notin \{0, 1\}$.

Bernoulli Random Variable

Definition

A discrete random variable $X \in \{0, 1\}$ is called a Bernoulli random variable with parameter $p \in (0, 1)$ (denoted as $X \sim \text{Bernoulli}(p)$), if the PMF of X is given as:

$$p_X(x) = \begin{cases} 1 - p & \text{if } x = 0 \\ p & \text{if } x = 1 \\ 0 & \text{elsewhere} \end{cases}$$

Note

- If $X \sim \text{Bernoulli}(p)$, the underlying Ω could be either countable or uncountable. However, X must partition Ω into two groups.

Bernoulli Random Variable

Example

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $\mathbf{P} : \mathcal{F} \to [0,1]$ be such that $\mathbf{P}([a,b]) = \mathbf{P}([a,b)) = b^2 - a^2$, where $0 \leq a < b \leq 1$. Let $X : \Omega \to \mathbb{R}$ be such that $X(\omega) = 0$ if $\omega \in [0, 0.5)$ and $X(\omega) = 1$ if $\omega \in [0.5, 1]$. Find the PMF of X.

Solution

$$\mathbf{P}(X = x) = \begin{cases} \mathbf{P}([0, 0.5)) & \text{if } x = 0 \\ \mathbf{P}([0.5, 1]) & \text{if } x = 1 \\ \mathbf{P}(\phi) & \text{elsewhere} \end{cases} = \begin{cases} 0.25 & \text{if } x = 0 \\ 0.75 & \text{if } x = 1 \\ 0 & \text{elsewhere} \end{cases}$$

- Hence, $X \sim \text{Bernoulli}(0.75)$.

Brushing Up on the Basics

Important Results

(a) (Finite Geometric Sum) If $q \in \mathbb{R} \setminus \{1\}$ and $n \in \mathbb{N}$, then $\sum_{x=1}^{n} q^{x-1} = \frac{1-q^n}{1-q}$.

(b) (Infinite Geometric Sum) If $|q| < 1$, then $\sum_{x=1}^{\infty} q^{x-1} = \frac{1}{1-q}$.

(c) (Binomial Expansion) If $n \in \mathbb{N}$ and $p, q \in \mathbb{R}$, then $(p+q)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$.

(d) (Exponential Series) If $\lambda \in \mathbb{R}$, then $e^\lambda = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$.

(e) (Exponential as a Limit) If $\lambda \in \mathbb{R}$, then $e^\lambda = \lim_{x\to\infty} \left(1 + \frac{\lambda}{x}\right)^x$.

(f) (Negative Binomial Series) If $r \in \mathbb{N}$ and $|q| < 1$, then

$$(1-q)^{-r} = \sum_{x=r}^{\infty} \binom{x-1}{r-1} q^{x-r}$$

Geometric Random Variable

Example

Consider a sequence of infinite coin tosses. Assume that the outcome of each toss is H with probability p and T with probability $1-p$, independent of outcomes of other tosses. Let X denote the number of tosses needed to get the first H.

- Define A_i to be the event that the outcome of the ith toss is H.
- Clearly, $\mathbf{P}(X=n) = \mathbf{P}(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n)$.
- Since $\mathbf{P}(A_i) = p, \forall i \in \mathbb{N}$ and A_i's are independent, $\mathbf{P}(X=n) = p(1-p)^{n-1}$.
- The CCDF is $\mathbf{P}(X > n) = p\sum_{x=n+1}^{\infty}(1-p)^{x-1} = (1-p)^n$.

Geometric Random Variable

Definition

A discrete random variable $X \in \{1, 2, \cdots\}$ is called a geometric random variable with parameter $p \in (0, 1)$ (denoted as $X \sim \text{Geom}(p)$) if its PMF is given as follows.

$$p_X(x) = \begin{cases} p(1-p)^{x-1} & \text{if } x \in \{1, 2, \cdots\} \\ 0 & \text{elsewhere} \end{cases}$$

Note

- The p_X function stated above is a valid PMF since $p_X(x) \in [0, 1], \forall x \in \mathbb{R}$ and $\sum_{x=1}^{\infty} p_X(x) = p \sum_{x=1}^{\infty} (1-p)^{x-1} = 1$.

Memoryless Property

Example

Consider the infinite coin toss model discussed before. Given that the first n tosses turn out to be all Ts, find the conditional probability that the next r tosses will also yield all Ts.

Solution

- Let X be the number of tosses needed to get the first H. The desired probability is:

$$\begin{aligned}\mathbf{P}(X > n + r \mid X > n) &= \frac{\mathbf{P}(X > n, X > n + r)}{\mathbf{P}(X > n)} \\ &= \frac{\mathbf{P}(X > n + r)}{\mathbf{P}(X > n)} = \frac{(1-p)^{n+r}}{(1-p)^{n}} = (1-p)^{r} = \mathbf{P}(X > r)\end{aligned}$$

Memoryless Property

Interpretation

Note that the occurrence of all Ts in the first n tosses does not affect the probability of occurrence of all Ts in the next r tosses. Therefore, in some sense, the memory of the first n tosses is not retained in the next r tosses.

Definition

A discrete random variable $X \in \{1, 2, \cdots\}$ is said to obey the memoryless property if $\mathbf{P}(X > n + r \mid X > n) = \mathbf{P}(X > r)$, where $n, r \in \{0, 1, \cdots\}$.

- We can show that if a discrete random variable is memoryless, then it must be a geometric random variable.

Binomial Random Variable

Example

Consider n coin tosses. Assume that the outcome of each toss is H with probability p and T with probability $1-p$, independent of the outcomes of other tosses. Let X be the number of Hs obtained. Our goal is to compute $\mathbf{P}(X=k)$, where $k \in \{0, \cdots, n\}$.

- Let $\mathcal{I}_k = \{i_1, \cdots, i_k\}$ be the indices of the coin tosses whose outcomes are H, and $\mathcal{I}_k^c$ be the indices of the coin tosses whose outcomes are T.
- Let A_i be event that the outcome of the ith toss is H. Clearly, A_i's are independent and $\mathbf{P}(A_i) = p$. Hence, $\mathbf{P}((\cap_{i\in\mathcal{I}_k} A_i) \cap (\cap_{j\in\mathcal{I}_k^c} A_j^c)) = p^k(1-p)^{n-k}$.
- The number of choices for $\mathcal{I}_k$ is $\binom{n}{k}$. Thus, $\mathbf{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$.

Binomial Random Variable

Definition

A discrete random variable $X \in \{0, 1, \cdots, n\}$ is called a binomial random variable with parameters (n, p), where $n \in \mathbb{N}$, $p \in (0, 1)$ (denoted as $X \sim \text{Binomial}(n, p)$) if its PMF is:

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & \text{if } x \in \{0, 1, \cdots, n\} \\ 0 & \text{elsewhere} \end{cases}$$

Note

- The above p_X function is a valid PMF since $p_X(x) \geq 0, \forall x \in \mathbb{R}$ and $\sum_{x=0}^{n} p_X(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (p + 1 - p)^n = 1$.

Poisson Random Variable

Definition

A discrete random variable $X \in \{0, 1, \cdots\}$ is called a Poisson random variable with parameter $\lambda > 0$ (denoted as $X \sim \text{Poisson}(\lambda)$) if its PMF is given as:

$$p_X(x) = \begin{cases} e^{-\lambda}\frac{\lambda^x}{x!} & \text{if } x \in \{0, 1, \cdots\} \\ 0 & \text{elsewhere} \end{cases}$$

Note

- The function p_X stated above is a valid PMF since $p_X(x) \geq 0, \forall x \in \mathbb{R}$ and $\sum_{x=0}^{\infty} p_X(x) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = 1$.

Poisson Random Variable

Example

The number of photons detected by a detector per minute, X, is a Poisson random variable with parameter $\lambda = 2$. Find the probability that 2 or more photons are detected per minute.

Solution

- The required probability is:

$$\mathbf{P}(X \geq 2) = 1 - \sum_{x=0}^{1} \mathbf{P}(X = x) = 1 - \sum_{x=0}^{1} e^{-2} \frac{2^x}{x!} = 0.594$$

Limiting Binomial is Poisson

Proposition 8.1

Fix $x \in \{0, 1, \cdots\}$ and $\lambda \in (0, \infty)$. If $np = \lambda$ or $p = \lambda/n$, then

$$\lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} = e^{-\lambda} \frac{\lambda^x}{x!}$$

Proof of Proposition 8.1

- Let $n > x$. We have the following.

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} = \frac{n(n-1)\cdots(n-x+1)}{x!}$$

Limiting Binomial is Poisson

Proof of Proposition 8.1 (Continued)

$$\begin{aligned}
\lim_{n\to\infty} \binom{n}{x} p^x &= \lim_{n\to\infty} \frac{n(n-1)\cdots(n-x+1)}{x!}\left(\frac{\lambda}{n}\right)^x \\
&= \frac{\lambda^x}{x!}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) = \frac{\lambda^x}{x!}
\end{aligned}$$

$$\lim_{n\to\infty}(1-p)^n = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\lim_{n\to\infty}(1-p)^{-x} = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1$$

Limiting Binomial is Poisson

Example

A certain machine has 100 components, each of which fails independently of the others with probability 0.05. Find the approximate probability that 3 or more components fail.

Solution

- Let X denote the number of components that fail. Clearly, $X \sim \text{Binomial}(100, 0.05)$. We can approximate it as $X \sim \text{Poisson}(5)$. Hence,

$$\mathbf{P}(X \geq 3) = 1 - \sum_{x=0}^{2} \mathbf{P}(X = x) \approx 1 - \sum_{x=0}^{2} e^{-5} \frac{5^x}{x!} = 0.875$$

Negative Binomial Random Variable

Example

Consider a sequence of infinite coin tosses, where the outcome of each toss is H and T with probabilities p and $1-p$, respectively, independent of outcomes of other tosses. Let X be the number of tosses needed to get r Hs. Let us compute $\mathbf{P}(X=k), k \geq r$.

- For $\{X=k\}$ to happen, the kth toss must yield H and out of preceding $k-1$ tosses, exactly $r-1$ tosses must yield H.
- The probability that r specified tosses yield H and rest $k-r$ yield T is $p^r(1-p)^{k-r}$.
- Moreover, $r-1$ tosses can be chosen out of $k-1$ ones in $\binom{k-1}{r-1}$ ways.
- Hence, $\mathbf{P}(X=k) = \binom{k-1}{r-1}p^r(1-p)^{k-r}$.

Negative Binomial Random Variable

Definition

A discrete random variable $X \in \{r, r+1, \cdots\}$ is called a negative binomial random variable with parameters (r, p) (denoted as $X \sim \text{NB}(r, p)$) if its PMF is:

$$p_X(x) = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r} & \text{if } x \in \{r, r+1, \cdots\} \\ 0 & \text{elsewhere} \end{cases}$$

Note

- The function p_X stated above is a valid PMF since $p_X(x) \geq 0$, $\forall x \in \mathbb{R}$, and $\sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r (1-p)^{x-r} = p^r p^{-r} = 1$.

Binomial vs Negative Binomial

Proposition 8.2

If $X \sim \text{Binomial}(n, p)$, $Y \sim \text{NB}(r, p)$ are random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $r \leq n$, then $\mathbf{P}(X \leq r) = \mathbf{P}(Y \geq n)$ i.e.,

$$\sum_{x=0}^{r} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=n}^{\infty} \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

Proof of Proposition 8.2

- $\mathbf{P}(X \leq r) = \mathbf{P}(\text{at most } r \text{ heads out of } n \text{ tosses})$
- $\mathbf{P}(Y \geq n) = \mathbf{P}(\text{at least } n \text{ tosses to get } r \text{ heads})$