Probability and Stochastic Processes
Lecture 9: Continuous Random Variables

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Continuous Random Variables

Definition

A random variable X associated with a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called a continuous random variable if its induced probability measure $\mathbf{P}_X$ can be expressed as follows for some non-negative function $f_X : \mathbb{R} \to [0, \infty)$.

$$\mathbf{P}_X(B) = \int_B f_X(x) \mathrm{d}x, \quad \forall B \in \mathcal{B}(\mathbb{R})$$

- f_X is called the probability density function (PDF) of X.
- Since $\mathbf{P}_X$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we get

$$\mathbf{P}_X(\mathbb{R}) = \int_{\mathbb{R}} f_X(x) \mathrm{d}x = 1$$

Continuous Random Variables

Notes

(a) If $B = [a, b]$, then $\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)\mathrm{d}x, \; a < b$.

(b) If $B = [a, b)$, then $\mathbf{P}(a \le X < b) = \int_a^b f_X(x)\mathrm{d}x, \; a < b$.

(c) If $B = (a, b]$, then $\mathbf{P}(a < X \le b) = \int_a^b f_X(x)\mathrm{d}x, \; a < b$.

(d) If $B = (a, b)$, then $\mathbf{P}(a < X < b) = \int_a^b f_X(x)\mathrm{d}x, \; a < b$.

Notes

(e) If $B = (-\infty, a)$, then $\mathbf{P}(X < a) = \int_{-\infty}^a f_X(x)\mathrm{d}x$.

(f) If $B = (-\infty, a]$, then $F_X(a) = \mathbf{P}(X \le a) = \int_{-\infty}^a f_X(x)\mathrm{d}x$.

Continuous Random Variables

Notes

(a) If $B = \{a\}$, then $\mathbf{P}(X = a) = \int_a^a f(x)\mathrm{d}x = 0, \ \forall a \in \mathbb{R}$.

(b) Hence, F_X is both right and left continuous *everywhere*, unlike the discrete case.

(c) If F_X is differentiable at x, then $f_X(x) = F'_X(x)$.

Interpretation of PDF

- If Δa is small, then $\mathbf{P}(a \leq X \leq a + \Delta a) \approx f_X(a)\Delta a$.
- Note that $f_X(a)$ is itself NOT a probability value. However, $f_X(a)\Delta a$ is a probability value for small Δa. That is why f_X is called a probability *density* function.

Continuous Random Variables

Example

Let X be a continuous random variable associated with a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, whose PDF, f_X, is given below.

$$f_X(x) = \begin{cases} C(4x - 2x^2), & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find C.

(b) Find $\mathbf{P}(X > 1)$.

(c) Find the CDF F_X.

Continuous Random Variables

Solution

(a) We must have $\int_{\mathbb{R}} f(x)\mathrm{d}x = 1$. Note that

$$1 = \int_{\mathbb{R}} f_X(x)\mathrm{d}x = \int_0^2 C(4x - 2x^2)\mathrm{d}x = C\left\{2x^2 - \frac{2x^3}{3}\right\}_0^2 = \frac{8C}{3}$$

(b) We can compute $\mathbf{P}(X > 1)$ as follows.

$$\mathbf{P}(X > 1) = \int_1^2 \frac{3}{8}(4x - 2x^2)\mathrm{d}x = \frac{3}{8}\left\{2x^2 - \frac{2x^3}{3}\right\}_1^2 = \frac{1}{2}$$

Continuous Random Variables

Solution (Continued)

(c) The CDF can be computed as follows.

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\mathrm{d}y = \begin{cases} 0 & \text{if } x < 0 \\ \frac{3}{8}\left\{2x^2 - \frac{2x^3}{3}\right\} & \text{if } 0 \leq x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}$$

- Note that $\mathbf{P}(X > 1) = 1 - F_X(1) = \frac{1}{2}$.

Uniform Random Variable

Definition

A continuous random variable X is said to be uniformly distributed in (a, b), denoted as $X \sim \mathrm{Unif}(a, b)$, if its PDF is given as follows.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Notes

- Note that $f_X(x) \geq 0, \forall x \in \mathbb{R}$.
- Moreover, $\int_{\mathbb{R}} f_X(x)\mathrm{d}x = \int_a^b \frac{1}{b-a}\mathrm{d}x = 1$.

Uniform Random Variable

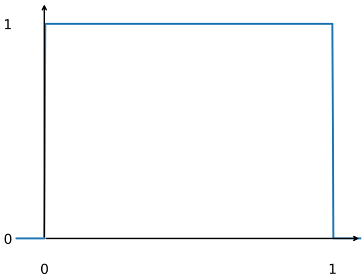
Notes

- The CDF of $X \sim \mathrm{Unif}(a, b)$ is given as follows.

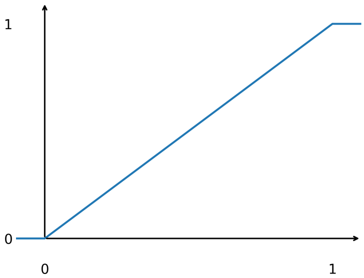
$$F_X(x) = \int_{-\infty}^{x} f_X(y)\mathrm{d}y = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } b \leq x \end{cases}$$

- Note that F_X takes values in $[0, 1]$ since $F_X(x) = \mathbf{P}(X \leq x)$.
- However, f_X can take value larger than 1. For example, if $X \sim \mathrm{Unif}(0, 0.2)$, then $f_X(x) = 5$, if $0 < x < 0.2$.

Uniform Random Variable



(a) PDF of $X \sim \mathrm{Unif}(0, 1)$



(b) CDF of $X \sim \mathrm{Unif}(0, 1)$

Uniform Random Variable

Example

Buses arrive at a specified stop at 15-minute intervals starting at 7 A. M., i.e., they arrive at $7, 7:15, 7:30, 7:45$, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and $7:30$, find the probability that he waits

(a) less than 5 minutes for a bus;

(b) more than 10 minutes for a bus.

Uniform Random Variable

Solution

- Assume that the passenger arrives X minutes past 7. The PDF of X is given below.

$$f_X(x) = \begin{cases} \frac{1}{30} & \text{if } 0 < x < 30 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Probability of waiting less than 5 minutes is

$$\mathbf{P}(10 < X < 15) + \mathbf{P}(25 < X < 30) = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

(b) Probability of waiting more than 10 minutes is

$$\mathbf{P}(0 < X < 5) + \mathbf{P}(15 < X < 20) = \frac{1}{3}$$

Exponential Random Variable

Definition

A continuous random variable X is called exponentially distributed with parameter $\lambda > 0$, denoted as $X \sim \exp(\lambda)$, if its PDF is given as follows.

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x), & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Notes

- Note that $f_X(x) \geq 0, \forall x \in \mathbb{R}$.
- Also, $\int_{\mathbb{R}} f_X(x) \mathrm{d}x = \int_0^\infty \lambda \exp(-\lambda x) \mathrm{d}x = 1$.

Exponential Random Variable

Cumulative Distribution Function (CDF)

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\mathrm{d}y = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \exp(-\lambda x) & \text{if } x \geq 0 \end{cases}$$

Memoryless Property

- Note that $\mathbf{P}(X > x) = \exp(-\lambda x)$ for $x > 0$. If $s, t > 0$, then

$$\mathbf{P}(X > s + t \mid X > s) = \frac{\mathbf{P}(X > s + t, X > s)}{\mathbf{P}(X > s)} = \frac{\mathbf{P}(X > s + t)}{\mathbf{P}(X > s)} = \mathbf{P}(X > t)$$

Exponential Random Variable

Example

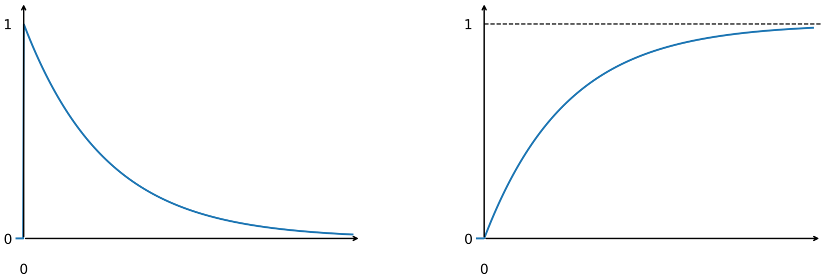
Let X denote the distance (in thousands of kilometers) a car travels before its battery wears out. Assume that $X \sim \exp(0.1)$. Assume that the car has already traveled 5000 km. If a person wishes to take an additional 5000-km trip, what is the probability that he or she will be able to complete it without having to replace the car battery?

Solution

- The desired probability is

$$\mathbf{P}(X > 10 \mid X > 5) = \mathbf{P}(X > 5) = \exp(-0.5) = 0.6065$$

Exponential Random Variable



(a) PDF of $X \sim \exp(1)$ (b) CDF of $X \sim \exp(1)$

Normal Random Variable

Definition

- A continuous random variable X is said to be normally distributed with parameters (μ, σ^2), denoted as $X \sim \mathcal{N}(\mu, \sigma^2)$, if its PDF is given as follows.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \ \forall x \in \mathbb{R}$$

- If $X \sim \mathcal{N}(0,1)$, then X is called a standard normal random variable.
- The CDF of $X \sim \mathcal{N}(0,1)$ is denoted as $\Phi(\cdot)$, which is defined below.

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy, \ \forall x \in \mathbb{R}$$

Normal Random Variable

Cumulative Distribution Function (CDF)

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then the CDF of X can be obtained as follows.

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) \mathrm{d}y$$

$$\overset{(a)}{=} \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- Equation (a) follows by substituting $z = (y-\mu)/\sigma$.

Normal Random Variable

Validating the PDF of Standard Normal Random Variable

- If $X \sim \mathcal{N}(0,1)$, and f_X is its PDF, then

$$\int_{\mathbb{R}} f_X(x)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \sqrt{\frac{2}{\pi}} \underbrace{\int_0^{\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x}_{J}$$

- It remains to show that $J = \sqrt{\pi/2}$. Note that J^2 can be expressed as follows.

$$J^2 = \int_0^{\infty} \int_0^{\infty} \exp\left(-\frac{x^2+y^2}{2}\right) \mathrm{d}x\mathrm{d}y$$

Normal Random Variable

Validating the PDF of Standard Normal Random Variable

- If $x = r\cos\theta, y = r\sin\theta$, then using the Jacobian, we get:

$$\mathrm{d}x\mathrm{d}y = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} \mathrm{d}r\mathrm{d}\theta = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} \mathrm{d}r\mathrm{d}\theta = r\mathrm{d}r\mathrm{d}\theta$$

- We obtain the following expression of J^2.

$$J^2 = \int_0^\infty \int_0^{\pi/2} r\exp\left(-\frac{r^2}{2}\right) \mathrm{d}r\mathrm{d}\theta = \frac{\pi}{2}$$

Normal Random Variable

Proposition 9.1

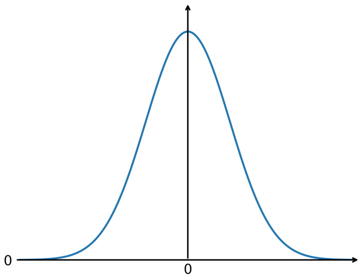
(a) $\Phi(-x) = 1 - \Phi(x), \ \forall x \in \mathbb{R}$.

(b) $\Phi(0) = 0.5$.

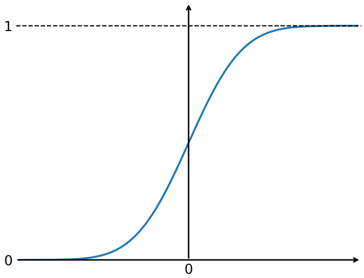
Proof of Proposition 9.1(a)

$$\Phi(-x) = \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y$$
$$= \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \mathrm{d}z = 1 - \Phi(x)$$

Normal Random Variable



(a) PDF of $X \sim \mathcal{N}(0, 1)$



(b) CDF of $X \sim \mathcal{N}(0, 1)$

Normal Random Variable

Example

If $X \sim \mathcal{N}(3, 9)$, then find $\mathbf{P}(|X-3| > 6)$.

Solution

$$\begin{aligned}
\mathbf{P}(|X-3| > 6) &= \mathbf{P}(X < -3) + \mathbf{P}(X > 9) \\
&= \mathbf{P}(X < -3) + 1 - \mathbf{P}(X \leq 9) \\
&= \Phi\left(\frac{-3-3}{3}\right) + 1 - \Phi\left(\frac{9-3}{3}\right) \\
&= \Phi(-2) + 1 - \Phi(2) = 2[1 - \Phi(2)] \approx 0.0456
\end{aligned}$$

Gamma Random Variable

Definition

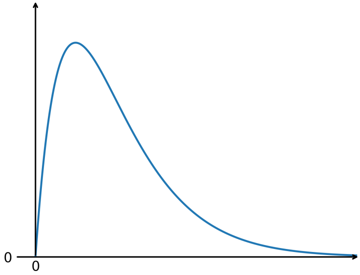
A continuous random variable X is said to be Gamma distributed with parameters (α, λ), where $\alpha, \lambda > 0$ if its PDF is expressed as follows, where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \exp(-y) \mathrm{d}y$.

$$f_X(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} \exp(-\lambda x)}{\Gamma(\alpha)}, & \text{if } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

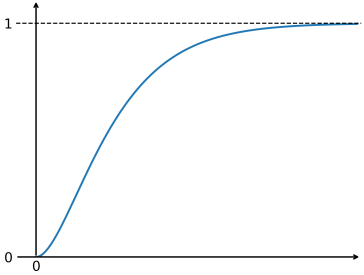
Notes

- We call α and λ the shape and the rate parameters, respectively.
- Note that the PDF of $\mathrm{Gamma}(1, \lambda)$ is the same as that of $\exp(\lambda)$.

Gamma Random Variable



(a) PDF of $X \sim \mathrm{Gamma}(2, 1)$



(b) CDF of $X \sim \mathrm{Gamma}(2, 1)$

Gamma Random Variable

Proposition 9.2

(a) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(b) $\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \forall\alpha > 0$.

Proof of Proposition 9.2(a)

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty y^{-\frac{1}{2}} \exp(-y)\mathrm{d}y$$
$$= \int_0^\infty \sqrt{2}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z = 2\sqrt{\pi}[1-\Phi(0)] = \sqrt{\pi}$$

Proof of Proposition 9.2(b)

- Using integration by parts, we obtain:

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^\infty y^\alpha \exp(-y)\mathrm{d}y \\ &= \{-y^\alpha \exp(-y)\}_0^\infty + \int_0^\infty \alpha y^{\alpha-1} \exp(-y)\mathrm{d}y \\ &= \alpha \int_0^\infty y^{\alpha-1} \exp(-y)\mathrm{d}y = \alpha\Gamma(\alpha)\end{aligned}$$

Gamma Random Variable

Proposition 9.3

(a) $\Gamma(1) = 1$.

(b) $\Gamma(n) = (n-1)!$, where $n \in \mathbb{N}$.

(c) $\Gamma\left(n + \frac{1}{2}\right) = \dfrac{(2n)!}{n!4^n}\sqrt{\pi}$, where $n \in \mathbb{N} \cup \{0\}$.

Proof of Proposition 9.3

(a) $\Gamma(1) = \int_0^\infty \exp(-y)\mathrm{d}y = 1$.

(b) $\Gamma(n) = (n-1)\Gamma(n-1) = \cdots = (n-1)!\Gamma(1) = (n-1)!$

Proof of Proposition 9.3 (Continued)

(c) Note the following.

$$
\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \\
&= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{(2n-1)(2n-3)\cdots 1}{2^n}\sqrt{\pi} \\
&= \frac{(2n)!}{2^n(2n)(2n-2)\cdots 2}\sqrt{\pi} = \frac{(2n)!}{n!4^n}\sqrt{\pi}
\end{aligned}
$$

Gamma Random Variable

Connection with Poisson Process

Let the number of photons being detected by a detector in an interval of length t is Poisson distributed with parameter λt, where $\lambda > 0$ is a constant. If the random variable T denotes the time needed to detect n photons, find the PDF of T as a function of n.

Solution

- Let $N(t) \sim \text{Poisson}(\lambda t)$ be the number of photons detected in an interval of length t.
- Note that $\{T \leq t\}$ is identical to the event $\{N(t) \geq n\}$. Therefore,

$$F_T(t) = \mathbf{P}(T \leq t) = \mathbf{P}(N(t) \geq n) = \sum_{k=n}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^k}{k!}$$

Gamma Random Variable

Solution (Continued)

- Differentiating with respect to t, we get:

$$f_T(t) = \sum_{k=n}^{\infty} \lambda \exp(-\lambda t) \frac{(\lambda t)^{k-1}}{(k-1)!} - \sum_{k=n}^{\infty} \lambda \exp(-\lambda t) \frac{(\lambda t)^{k}}{k!}$$

$$= \sum_{k=n-1}^{\infty} \lambda \exp(-\lambda t) \frac{(\lambda t)^{k}}{k!} - \sum_{k=n}^{\infty} \lambda \exp(-\lambda t) \frac{(\lambda t)^{k}}{k!}$$

$$= \lambda \exp(-\lambda t) \frac{(\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda^n t^{n-1} \exp(-\lambda t)}{\Gamma(n)}$$

- Hence, $T \sim \text{Gamma}(n, \lambda)$.

Beta Random Variable

Definition

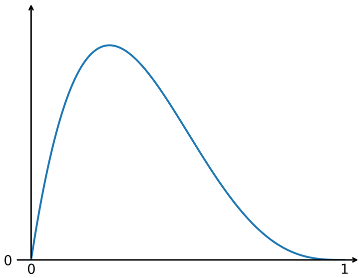
A continuous random variable X is called a Beta random variable with parameters (a, b) if its PDF is expressed as follows, where $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$, and $a, b > 0$.

$$f_X(x) = \begin{cases} \dfrac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

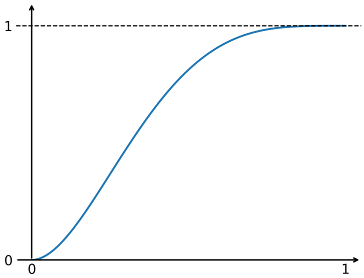
Notes

- If $a = b$, then the PDF is symmetric about $x = \frac{1}{2}$ axis.
- If $a = b = 1$, then the PDF reduces to that of the uniform distribution on $(0, 1)$.

Beta Random Variable



(a) PDF of $X \sim \mathrm{Beta}(2,4)$



(b) CDF of $X \sim \mathrm{Beta}(2,4)$

Beta Random Variable

Proposition 9.4

The following holds for any $a, b > 0$.

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Proof of Proposition 9.4

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty x^{a-1} y^{b-1} \exp(-(x+y)) \mathrm{d}x \mathrm{d}y$$

Beta Random Variable

Proof of Proposition 9.4 (Continued)

- Substituting $x = vt$ and $y = (1-v)t$, we get the following using the Jacobian.

$$\mathrm{d}x\mathrm{d}y = \begin{vmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \mathrm{d}v\mathrm{d}t = \begin{vmatrix} t & -t \\ v & 1-v \end{vmatrix} \mathrm{d}v\mathrm{d}t = t\mathrm{d}v\mathrm{d}t$$

- Note that $t = (x+y)$, and $v = x/(x+y)$. We get,

$$\Gamma(a)\Gamma(b) = \int_0^\infty t^{a+b-1} \exp(-t)\mathrm{d}t \int_0^1 v^{a-1}(1-v)^{b-1}\mathrm{d}v = \Gamma(a+b)B(a,b)$$

Practical Applications

- Normal: Noise modeling in wireless communications.
- Exponential: Modeling radioactive decay, packet arrival process, etc.
- Gamma: Modeling rainfall amount, insurance claims, etc.
- Beta: Conjugate prior in Bayesian inference statistics.
- There are many others we have not covered.

Other Distributions

Rayleigh Distribution

A simple model of wireless Base Station (BS) placement in a 2D plane is called a Poisson Point Process (PPP), where it is assumed that the number of BSs placed in a region of area A is Poisson distributed with parameter λA for some $\lambda > 0$. Assume that a user is located at the origin. If R denotes its distance to the closest BS, then find the PDF of R.

Solution

- $\{R > r\}$ is the event that there are no BSs within a circle of radius r centered at $\mathbf{0}$.
- Note that $1 - F_R(r) = \mathbf{P}(R > r) = \exp(-\lambda\pi r^2)$, where $r > 0$.
- Therefore, $f_R(r) = 2\pi\lambda r \exp(-\lambda\pi r^2)$, $r > 0$.