Probability and Stochastic Processes: Problem Set 1

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Q. 1: Define a binary operation Δ (called the symmetric difference operation) such that $A\Delta B = (A \setminus B) \cup (B \setminus A)$ for arbitrary sets A, B.

(a) Prove that $A\Delta B = (A \cup B) \setminus (A \cap B)$ for any A, B.
(b) Prove that $A\Delta B = B\Delta A$ for any A, B, i.e., Δ is commutative.
(c) Prove that $(A\Delta B)\Delta C = A\Delta(B\Delta C)$ for any A, B, C, i.e., Δ is associative.
(d) Prove that $A \cap (B\Delta C) = (A \cap B)\Delta(A \cap C)$ for any A, B, C, i.e., Δ is distributive over $\cap$.

Q. 2: Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of sets. Define another collection $\{B_i\}_{i=1}^{\infty}$ such that $B_1 = A_1$ and $B_i = A_i \setminus (A_1 \cup \cdots \cup A_{i-1})$, $\forall i \in \{2, 3, \cdots\}$.

(a) Prove that B_i's are mutually disjoint, i.e., $B_i \cap B_j = \phi$ if $i \neq j$.
(b) Establish that $\cup_{i\in I} A_i = \cup_{i\in I} B_i$, where $I \subset \mathbb{N}$. To prove this statement, you might have to use the well-ordering principle, which states that every non-empty $I \subset \mathbb{N}$ has a least element. One can utilize this result to show that $\cup_{i=1}^{n} A_i = \cup_{i=1}^{n} B_i$, $\forall n \in \mathbb{N}$, and $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$.

Consider an uncountable collection of sets $\{A_\alpha\}_{\alpha\in[0,1]}$ such that

$$A_\alpha = \begin{cases} \{a\} & \text{if } \alpha \in [0, 0.5] \\ \{a, b\} & \text{if } \alpha \in (0.5, 1] \end{cases}$$

Define another collection $\{B_\alpha\}_{\alpha\in[0,1]}$ such that $B_0 = A_0$ and $B_\alpha = A_\alpha \setminus \left(\cup_{x\in[0,\alpha)} A_x\right)$ for $\alpha \in (0, 1]$.

(c) Demonstrate that $\cup_{\alpha\in[0,1]} A_\alpha \neq \cup_{\alpha\in[0,1]} B_\alpha$. Why does the earlier proof break down in this case?

Q. 3: Let A and B be finite sets with cardinalities $|A| = p$ and $|B| = q$. Prove the following statements.

(a) The number of relations and functions from A to B are 2^{pq} and q^p, respectively.
(b) If $p \leq q$, then the number of injective functions from A to B is $q(q-1)\cdots(q-p+1)$.
(c) If $p \geq q$, then the number of surjective functions from A to B is

$$\sum_{k=0}^{q} (-1)^k \binom{q}{k} (q-k)^p$$

Q. 4: Let A and B be arbitrary non-empty sets. Prove the following statements.

(a) If there exists an injection $f : A \to B$, then there exists a surjection $g : B \to A$.
(b) If there exists a surjection $f : A \to B$, then there exists an injection $g : B \to A$. To establish this statement, you might have to apply the Axiom of Choice (AoC), which states that it is always possible to choose one element from each set from an arbitrary collection of non-empty sets.

Q. 5: Let $f : A \to B$ and $g : B \to C$ be arbitrary functions, and A, B, C are non-empty sets. We define the composite function $g \circ f : A \to C$ such that $(g \circ f)(x) = g(f(x))$, $\forall x \in A$. Prove the following statements.

(a) If f, g are both injective, then so is the composite function $g \circ f$. Therefore, $|A| \leq |B|$ and $|B| \leq |C|$ imply that $|A| \leq |C|$ for arbitrary non-empty sets A, B, C.
(b) If f, g are both surjective, then so is the composite function $g \circ f$.
(c) If f, g are both bijective, then so is the composite function $g \circ f$. Hence, $|A| = |B|$ and $|B| = |C|$ imply that $|A| = |C|$ for arbitrary non-empty sets A, B, C.
(d) The pre-image function of $g \circ f$ is related to those of f and g as $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Q. 6: Let A, B, C, D be (not necessarily finite) non-empty sets such that $|A| = |B|$ and $|C| = |D|$. Prove that $A \times C$ and $B \times D$ have the same cardinality.

Q. 7: Show that the following infinite sets have the same cardinality.

(a) $(0, 1]$, $[0, 1)$, $(0, 1)$, and $[0, 1]$.

(b) $[0, 1)$ and $[0, \infty)$.

(c) $(0, 1)$ and $\mathbb{R}$.

(d) $[0, 1]$ and $2^{\mathbb{N}}$ (the collection of all subsets of $\mathbb{N}$).

(e) $(0, 1)$ and $(0, 1) \times (0, 1)$.

(f) $\mathbb{R}$ and $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n times), where $n \in \mathbb{N}$.

Q. 8: Prove that the following sets are countable.

(a) The collection of pair of natural numbers $\mathbb{N} \times \mathbb{N}$.

(b) The set of all integers $\mathbb{Z}$.

(c) The cartesian product $\mathbb{Z} \times \mathbb{N}$.

(d) The set of rational numbers $\mathbb{Q}$.

Q. 9: Prove that a countable union of countable sets is countable. In particular, if $\{A_i\}_{i \in \mathbb{N}}$ is such that A_i is countable $\forall i \in \mathbb{N}$, then show that $\cup_{i \in \mathbb{N}} A_i$ is countable.

Q. 10: The density of rationals ($\mathbb{Q}$) in $\mathbb{R}$ states that if $x, y \in \mathbb{R}$ such that $x < y$, then there exists a $q \in \mathbb{Q}$ such that $x < q < y$. Utilize this property to show that if $\{(x_\alpha, y_\alpha)\}_{\alpha \in I}$ is an *arbitrary* collection of open intervals, where $x_\alpha, y_\alpha \in \mathbb{R}$, and $x_\alpha < y_\alpha$, $\forall \alpha \in I$, then there exists a *countable* $J \subset I$ such that

$$\bigcup_{\alpha \in I} (x_\alpha, y_\alpha) = \bigcup_{\alpha \in J} (x_\alpha, y_\alpha)$$