Probability and Stochastic Processes: Problem Set 2

Instructor: Prof. Washim Uddin Mondal
Department of Electrical Engineering, IIT Kanpur

Q. 1: Let $\Omega = (0, 1]$, and $\mathcal{F}$ be the collection of subsets of $(0, 1]$ that can be expressed as $(a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n]$, where $0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \leq 1$, for some $n \in \mathbb{N}$, and the null set ϕ.

(a) Show that $\mathcal{F}$ is an algebra of Ω.

(b) Show that $\mathcal{F}$ is not a σ-algebra of Ω.

(c) Show that $\sigma(\mathcal{F}) = \mathcal{B}((0, 1])$.

Q. 2: Let Ω be non-empty. Prove that $\mathcal{F} = \{A \subset \Omega \mid A \text{ is countable or } A \text{ is co-countable}\}$ is a σ-algebra of Ω. It is to be noted that a set A is called a co-countable set if A^c is countable.

Q. 3: Let $\mathcal{S} = \{\{x\} | \, x \in \mathbb{R}\}$ be the collection of all singleton subsets of $\mathbb{R}$.

(a) Show that $\sigma(\mathcal{S}) = \{A \subset \mathbb{R} \mid A \text{ is countable or } A^c \text{ is countable}\}$.

(b) Prove that $\sigma(\mathcal{S})$ is a proper subset of the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

Q. 4: Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Show that $\mathcal{G} = \{A \in \mathcal{F} \mid \mathbf{P}(A) \in \{0, 1\}\}$ is a σ-algebra of Ω.

Q. 5: Let $\{\mathcal{F}_i\}_{i=1}^{\infty}$ be a sequence of algebras of Ω such that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$, $\forall i \in \mathbb{N}$. Prove that $\cup_{i \in \mathbb{N}} \mathcal{F}_i$ is also an algebra of Ω. Explain why the proof cannot be extended to σ-algebras.

Q. 6: Let $\mathcal{P}_1 = \{\phi, \{a, b\}\}$, $\mathcal{P}_2 = \{\{b, c\}\}$, and $\Omega = \{a, b, c\}$. Prove the following statements.

(a) $\mathcal{P}_1$ and $\mathcal{P}_2$ are π-systems of Ω.

(b) $\mathcal{P}_1 \cup \mathcal{P}_2$ is not a π-system of Ω. Hence, π-systems are not closed under union.

(c) $\mathcal{P}_1 \cap \mathcal{P}_2$ is not a π-system of Ω. Hence, π-systems are not closed under intersection.

(d) $\mathcal{P}_1$ is not a σ-algebra. Therefore, not every π-system is a σ-algebra.

(e) Every σ-algebra is a π-system.

Q. 7: Let A, B, C be three events associated with the probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

(a) If $\mathbf{P}(A) = \mathbf{P}(B) = 1$, then show that $\mathbf{P}(A \cap B) = 1$.

(b) If $\mathbf{P}(A|B) = 1$, then show that $\mathbf{P}(B^c|A^c) = 1$.

(c) If $\mathbf{P}(A), \mathbf{P}(B) > 0$ and A, B are independent, then show that they are NOT disjoint.

(d) Prove the following equality assuming $\mathbf{P}(B|C), \mathbf{P}(C|B), \mathbf{P}(A), \mathbf{P}(B), \mathbf{P}(C) > 0$.

$$\frac{\mathbf{P}(A|C)}{\mathbf{P}(B|C)} = \frac{\mathbf{P}(A)}{\mathbf{P}(B)} \frac{\mathbf{P}(C|A)}{\mathbf{P}(C|B)}$$

(e) Establish that $\mathbf{P}(A|B) = \mathbf{P}(A|B \cap C)\mathbf{P}(C|B) + \mathbf{P}(A|B \cap C^c)\mathbf{P}(C^c|B)$. Assume that $\mathbf{P}(B \cap C) > 0$ and $\mathbf{P}(B \cap C^c) > 0$.

(f) Show that $\mathbf{P}(A \cap B|A) \geq P(A \cap B|A \cup B)$. Assume that $\mathbf{P}(A) > 0$.

Q. 8: Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. If A, B, C are independent $\mathcal{F}$-measurable sets, then establish the following statements.

(a) $A^c \cap B$ and C are independent.

(b) A and $B \cup C$ are independent.

Q. 9: A fair coin is tossed twice. The underlying sample space is $\Omega = \{HH, HT, TH, TT\}$, and all outcomes are equally likely. Define three events A, B, C as follows.

- A : the result of the first toss is H i.e., $A = \{HH, HT\}$.
- B : the result of the second toss is H, i.e., $B = \{TH, HH\}$.
- C : the results of both the tosses are the same, i.e., $C = \{HH, TT\}$.

Show the following results.

(a) A, B, C are pairwise independent.

(b) A, B, C are NOT mutually independent.

The above exercise shows that, in general, pairwise independence does not imply mutual independence.

Q. 10: Let $\mathbf{P}_1, \mathbf{P}_2$ be probability measures associated with a measurable space $(\Omega, \mathcal{F})$, where $\Omega = \{1, 2, 3, 4\}$, and $\mathcal{F} = 2^{\Omega}$, such that

$$\mathbf{P}_1(\{1\}) = \frac{1}{4},\ \mathbf{P}_1(\{2\}) = \frac{1}{4},\ \mathbf{P}_1(\{3\}) = \frac{1}{4},\ \mathbf{P}_1(\{4\}) = \frac{1}{4},$$
$$\mathbf{P}_2(\{1\}) = \frac{1}{8},\ \mathbf{P}_2(\{2\}) = \frac{3}{8},\ \mathbf{P}_2(\{3\}) = \frac{1}{8},\ \mathbf{P}_2(\{4\}) = \frac{3}{8}$$

Let $\mathcal{S} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. Show the following results.

(a) $\mathbf{P}_1$ and $\mathbf{P}_2$ agree on $\mathcal{S}$.

(b) $\mathcal{F} = \sigma(\mathcal{S})$.

(c) $\mathcal{S}$ is not a π-system.

The above exercise shows that two probability measures agreeing on a collection $\mathcal{S}$ of subsets of Ω may not agree on the σ-algebra generated by $\mathcal{S}$. However, if $\mathcal{S}$ satisfies additional structural constraints (e.g., if $\mathcal{S}$ is a π-system), then two probability measures agreeing on $\mathcal{S}$ must agree on $\sigma(\mathcal{S})$ (due to Dynkin's Extension Theorem).

Q. 11: A box contains 5 red, 6 blue, and 8 green balls. If 3 balls are randomly selected without replacement, what is the probability that each of the balls will be:

(a) of the same color?

(b) of different colors?

Repeat the above exercises under the assumption that the balls are chosen with replacement.

Q. 12: From a group of 3 freshmen, 4 sophomores, 4 juniors, and 3 seniors, a committee of size 4 is randomly selected. Find the probability that the committee will consist of

(a) 1 from each class.

(b) 2 sophomores and 2 juniors.

(c) only sophomores or juniors.

Q. 13: Five people, designated as A, B, C, D, E, are arranged in linear order. Assuming that each possible order is equally likely, what is the probability that

(a) there is exactly one person between A and B?

(b) there are exactly two people between A and B?

(c) there are three people between A and B?

Q. 14: A pair of dice is rolled until a sum of either 3 or 5 appears. Find the probability that a 5 occurs first.

Q. 15: A box has b black and r red balls. A ball is drawn at random. When it is returned to the box, c additional balls of the same color are added. Suppose that we draw another ball. Prove that the probability that the first ball was black, given that the second ball drawn was red, is $b/(b + r + c)$.

Q. 16: A simplified model for the movement of the price of a stock supposes that on each day the stock's price either moves up 1 unit with probability p or moves down 1 unit with probability $1 - p$. The changes on different days are assumed to be independent.

(a) What is the probability that after 2 days the stock will be at its original price?

(b) What is the probability that after 3 days the stock's price will have increased by 1 unit?

(c) Given that after 3 days the stock's price has increased by 1 unit, what is the probability that it went up on the first day?

Q. 17: A red die, a blue die, and a yellow die (all six-sided) are rolled. We are interested in the probability that the number appearing on the blue die is less than that appearing on the yellow die, which is less than that appearing on the red die. That is, with B, Y, and R denoting, respectively, the number appearing on the blue, yellow, and red die, we are interested in the probability of the event $B < Y < R$.

(a) What is the probability that no two of the dice land on the same number?

(b) Given that no two of the dice land on the same number, what is the probability that $B < Y < R$?

(c) What is the probability that $B < Y < R$?

Q. 18: Suppose that n balls are to be put into r boxes, where each box can contain n balls. If all outcomes are equally likely, obtain the probability that no box remains empty.

Q. 19: Let A and B be two arbitrary subsets of $S = \{1, \cdots, n\}$. If all 2^n subsets of S are equally likely, show that the probability of the event $A \subset B$ is $(3/4)^n$.

Q. 20: A pond contains 3 distinct species of fish, which we will call the Red, Blue, and Green fish. There are r Red, b Blue, and g Green fish. Suppose that the fish are removed from the pond in a random order. What is the probability that the Red fish are the first species to become extinct in the pond?