Probability and Stochastic Processes: Problem Set 3

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Q. 1: Let $(\Omega, \mathcal{F})$ be a measurable space, and $X : \Omega \to \mathbb{R}$ be arbitrary. Let $\mathcal{P}$ denote an arbitrary collection of subsets of $\mathbb{R}$ such that $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R})$, and $X^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{P}$. Define the following collection.

$$\mathcal{L} = \left\{ B \subset \mathbb{R} \mid X^{-1}(B) \in \mathcal{F} \right\}$$

(a) Argue that $\mathcal{P} \subset \mathcal{L}$.
(b) Prove that $\mathcal{L}$ is a σ-algebra of $\mathbb{R}$.
(c) Use the above results to prove that $\sigma(\mathcal{P}) \subset \mathcal{L}$.
(d) Use the above result to prove that X is a random variable on $(\Omega, \mathcal{F})$, i.e., X is $\mathcal{F}$-measurable.
(e) Use the above result to prove that if $X^{-1}((a, \infty)) \in \mathcal{F}, \forall a \in \mathbb{R}$, then X is $\mathcal{F}$-measurable.
(f) Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Use the above result to prove that if $\{x \in \mathbb{R} \mid f(x) > a\}$ is a Borel set $\forall a \in \mathbb{R}$, then f is a Borel-measurable function.
(g) Use the above result to show that if $f : \mathbb{R} \to \mathbb{R}$ is a monotonic function, then f is Borel-measurable.

Q. 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function with countable number of discontinuities. Let C and D denote the sets of points where f is continuous and discontinuous, respectively. Clearly, $C \cap D = \phi$ and $C \cup D = \mathbb{R}$, i.e., $\{C, D\}$ is a partition of $\mathbb{R}$. Fix an $a \in \mathbb{R}$, and define $A = \{x \in \mathbb{R} \mid f(x) > a\}$.

(a) Use the $\epsilon - \delta$ definition of continuity to prove that if $y \in A \cap C$, then there exists $\delta_y > 0$ such that

$$(y - \delta_y, y + \delta_y) \subset A$$

(b) Use the above result to prove that there exist positive real numbers $\{\delta_y\}_{y \in A \cap C}$ such that

$$A = \left(\bigcup_{y \in A \cap C} (y - \delta_y, y + \delta_y) \right) \bigcup \Big(A \cap D \Big)$$

(c) Use the above result and **Q. 10** of P. Set 1 to show that there is a *countable* $I \subset A \cap C$ such that

$$A = \left(\bigcup_{y \in I} (y - \delta_y, y + \delta_y) \right) \bigcup \Big(A \cap D \Big)$$

(d) Use the above result to argue that A is a Borel set, i.e., $A \in \mathcal{B}(\mathbb{R})$. Explain the importance of part (c) to establish this result. In other words, explain why the result of part (b) is not sufficient to conclude the Borel-measurability of A.
(e) Use the above result and **Q. 1**(f) to conclude that f is a Borel-measurable function. Which part of the proof would break down if the number of discontinuities of f is uncountable?

Q. 3: Let $\Omega = \{a, b, c, d\}$, and $X : \Omega \to \mathbb{R}$ be such that $X(a) = X(b) = -1$, $X(c) = 0$, and $X(d) = 1$.

(a) Find $\sigma(X)$, i.e., the σ-algebra generated by X.
(b) Is X a random variable on the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F} = \{\phi, \Omega, \{a, b\}, \{c, d\}\}$?
(c) What is the minimum size of $\mathcal{F}$ that makes X a random variable on the measurable space $(\Omega, \mathcal{F})$?
(d) Define $Y : \Omega \to \mathbb{R}$ such that $Y(\omega) = X(\omega) + 2, \forall \omega \in \Omega$. Obtain $\sigma(Y)$.
(e) What is the relation between $\sigma(Y)$ and $\sigma(X)$?
(f) Define $Z : \Omega \to \mathbb{R}$ such that $Z(\omega) = X^2(\omega), \forall \omega \in \Omega$. Obtain $\sigma(Z)$.
(g) What is the relation between $\sigma(Z)$ and $\sigma(X)$?

Q. 4: Explain why the following functions of the form $f : \mathbb{R} \to \mathbb{R}$ must be Borel-measurable.

(a) $f(x) = \sin(\pi x), \forall x \in \mathbb{R}$.

(b) $f(x) = \lfloor x \rfloor, \forall x \in \mathbb{R}$, where $\lfloor x \rfloor$ is the largest integer not exceeding x (also known as the floor of x). For example, $\lfloor 3.14 \rfloor = 3$, $\lfloor -2.56 \rfloor = -3$, etc.

(c) The Dirichlet function, defined as follows, where $\mathbb{Q}$ is the set of rational numbers.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

An interesting property of the Dirichlet function is that it is discontinuous everywhere, i.e., it has uncountably many discontinuities.

Q. 5: Let $(\Omega, \mathcal{F})$ be a measurable space, $X : \Omega \to \mathbb{R}$ be arbitrary, and $f : \mathbb{R} \to \mathbb{R}$ be Borel-measurable.

(a) Prove that $\sigma(f(X)) \subset \sigma(X)$.

(b) Is the condition that f is Borel-measurable necessary? In other words, if f is an arbitrary function which may not be Borel-measurable, would the proof still work?

(c) As a corollary, prove that if X is a random variable on $(\Omega, \mathcal{F})$, then so is $f(X)$.

(d) Prove that if $g : \mathbb{R} \to \mathbb{R}$ is Borel-measurable, then so is the composite function $f \circ g : \mathbb{R} \to \mathbb{R}$.

Q. 6: Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, and $\mathbf{P} : \mathcal{F} \to [0, 1]$ be such that $\mathbf{P}([a, b]) = \mathbf{P}((a, b]) = \mathbf{P}([a, b)]) = \mathbf{P}((a, b)) = b - a$, where $0 \leq a < b \leq 1$. Define $X : \Omega \to \mathbb{R}$ as

$$X(\omega) = \begin{cases} \omega & \text{if } \omega \in [0, 0.5) \\ 1 & \text{if } \omega \in [0.5, 1] \end{cases}$$

(a) Obtain an expression of $X^{-1}(B)$, where $B \in \mathcal{B}(\mathbb{R})$. Utilize this result to argue that X is a random variable on $(\Omega, \mathcal{F})$, i.e., X is $\mathcal{F}$-measurable.

(b) Obtain the cumulative distribution function (CDF) of X, and show that X is neither a discrete nor a continuous random variable.

(c) Let $Y : \Omega \to \mathbb{R}$ be such that $Y(\omega) = \sqrt{\omega}, \forall \omega \in [0, 1]$. Prove that Y is a random variable on $(\Omega, \mathcal{F})$, i.e., Y is a $\mathcal{F}$-measurable function. Notice that it might be difficult to directly show that $Y^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}(\mathbb{R})$. What other strategy would you use to establish this result?

Q. 7: The cumulative distribution function (CDF) of a discrete random variable X is given as follows.

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.5 & \text{if } 0 \leq x < 1 \\ 0.6 & \text{if } 1 \leq x < 2 \\ 0.8 & \text{if } 2 \leq x < 3 \\ 0.9 & \text{if } 3 \leq x < 3.5 \\ 1 & \text{if } x \geq 3.5 \end{cases}$$

(a) Find the probability mass function (PMF) of X.

(b) Find the probability that $1.2 < X \leq 3$.

(c) Compute $\mathbf{E}[X]$ and $\mathrm{Var}(X)$.

(d) Use the Law of the Unconscious Statistician (LOTUS) to obtain the expected absolute mean deviation value $\mathbf{E}[|X - \mu|]$, where $\mu = \mathbf{E}[X]$.

(e) Use the LOTUS to compute the expectation $\mathbf{E}[\mathrm{sgn}(X - \mu)]$, where $\mu = \mathbf{E}[X]$ and

$$\mathrm{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(f) Define $Y = \mathrm{sgn}(X - \mu)$ where $\mu = \mathbf{E}[X]$. Calculate the PMF of Y.

(g) Using the above result, compute $\mathbf{E}[Y]$ and compare it with the result obtained in part (e).

Q. 8: Let X be an arbitrary random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Comment whether the following functions must be left- and/or right-continuous with appropriate reasoning.

(a) $F_1 : \mathbb{R} \to [0, 1]$ such that $F_1(x) = \mathbf{P}(X < x)$.

(b) $F_2 : \mathbb{R} \to [0, 1]$ such that $F_2(x) = \mathbf{P}(X > x)$.

(c) $F_3 : \mathbb{R} \to [0, 1]$ such that $F_3(x) = \mathbf{P}(X \geq x)$.

Q. 9: Let F_X be the cumulative distribution function (CDF) of the random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The goal of this exercise is to show that F_X can have at most countably many discontinuities. Towards this objective, define a set D_n as follows, where $n \in \mathbb{N}$.

$$D_n = \left\{ x \in \mathbb{R} \;\middle|\; F_X(x) - F_X(x^-) \geq \frac{1}{n} \right\}$$

where $F_X(x^-) = \lim_{h \to 0^+} F_X(x - h)$.

(a) Recall that $\mathbf{P}(X = x) = F_X(x) - F_X(x^-)$. Use this result to show that D_n is finite, $\forall n \in \mathbb{N}$.

(b) Note that the set of discontinuities of F_X, indicated as D, can be expressed as follows.

$$D = \left\{ x \in \mathbb{R} \mid F_X(x) - F_X(x^-) > 0 \right\}$$

How are D and the collection $\{D_n\}_{n \in \mathbb{N}}$ related? Use this result to show that D is countable.

Q. 10: Let X be a Poisson random variable with parameter λ.

(a) Show that $\mathbf{E}[X^n] = \lambda \mathbf{E}[(X + 1)^{n-1}]$ where $n \geq 1$.

(b) Use the above result to compute $\mathbf{E}[X^3]$.

Q. 11: Let X be a Binomial random variable with parameters n and p. Show that

$$\mathbf{E}\left[\frac{1}{X+1}\right] = \frac{1 - (1-p)^{n+1}}{(n+1)p}$$

Q. 12: Assume that the number of times a person contracts a cold in a given year is a Poisson random variable with parameter $\lambda = 5$. Suppose that a new drug has been marketed that reduces the Poisson parameter to $\lambda = 3$ for 75 percent of the population. For the other 25 percent of the population, the drug has no appreciable effect on colds. If an individual tries the drug for a year and has 2 colds in that time, how likely is it that the drug is beneficial for him or her?

Q. 13: Five men and 5 women are ranked according to their scores on an examination. Assume that no two scores are alike and all $10!$ possible rankings are equally likely. Let X denote the highest ranking achieved by a woman. (For instance, $X = 1$ if the top-ranked person is a female).

(a) Find the PMF of X.

(b) Find $\mathbf{E}[X]$.

Q. 14: Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X be the number of times player 1 wins. Find $\Pr(X = i)$, $i = 0, 1, 2, 3, 4$.

Q. 15: Two teams A and B play a series of games until one of them wins $k \geq 1$ games. Let us assume that outcomes of each of these games are independent of each other. Moreover, in each game, the team A wins with probability p and loses with probability $1 - p$. Let X denotes the number of games that are played.

(a) What is the range of values that X can take?

(b) Find the PMF of X. Verify that it is a valid PMF for $k = 1, 2$.

Q. 16: An unknown number (say N) of animals inhabit a certain region. To obtain information about the size of the population, ecologists often perform the following experiment: They first catch m number of these animals, mark them in some manner, and release them. After allowing the marked animals time to disperse throughout the region, a new catch of size $n < m$ is made. Let X denote the number of marked animals in this second capture. Assume that the population of animals in the region remained fixed between the time of the two catches and that each time an animal was caught, it was equally likely to be any of the remaining uncaught animals. We shall assume that $N > m + n$ since the population size is likely to be very large.

(a) Compute $\mathbf{P}(X = i)$ where $\mathbf{P}$ denotes the underlying probability measure and $i \in \{0, 1, \cdots, n\}$. The obtained PMF is called a Hypergeometric distribution with parameters (N, m, n).

(b) Show that for any integer $k \geq 1$, we have the following equality.

$$\mathbf{E}[X^k] = \frac{mn}{N}\mathbf{E}\left[(Y+1)^{k-1}\right]$$

where Y has a similar distribution as that of X but with parameters $(N-1, m-1, n-1)$.

(c) If $N = 20, m = 10, n = 4$, find the variance of X using the above relation.

Q. 17: Each night, different meteorologists give us the probability that it will rain the next day. To judge how well these people predict, we will score each of them as follows: If a meteorologist says that it will rain with probability p, then he or she will receive a score of

$$X = \begin{cases} 1 - (1-p)^2 & \text{if it rains} \\ 1 - p^2 & \text{otherwise} \end{cases}$$

We will then track scores over a given time period and conclude that the meteorologist with the highest average score is the best predictor of the weather. Suppose now that a given meteorologist is aware of our scoring mechanism and wants to maximize their expected score. If this person truly believes that it will rain tomorrow with probability p_0, what value of p should he or she assert so as to maximize the expected score?

Q. 18: Reason whether the following function can be a valid density function for some choice of the constant C. If yes, then find the constant, C.

$$f(x) = \begin{cases} C(2x - x^2), & \text{if } 0 \leq x \leq 2.5, \\ 0 & \text{elsewhere} \end{cases}$$

Q. 19: You arrive at a bus stop at 10 a.m., knowing that the bus will arrive at some time uniformly distributed between 10 a.m. and $10:30$ a.m.

(a) What is the probability that you will have to wait longer than 10 minutes?

(b) If, at $10:15$ a.m., the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

(c) What can we conclude about the memoryless property of the uniform distribution from the result (b)?

Q. 20: The speed of a molecule in a uniform gas at equilibrium is a random variable whose probability density function is given by

$$f(x) = \begin{cases} ax^2 \exp(-bx^2), & \text{if } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

where a, b are positive constants.

(a) Evaluate a in terms of b.

(b) Compute the average speed of a molecule as a function of b.

Q. 21: A fire station is to be built along a road of length A. If fire occurs at point X that is uniformly located on $(0, A)$, then how should the location of the station $a \in (0, A)$ be chosen to minimize the expected distance from the fire $\mathbf{E}[|X - a|]$? Here $|\cdot|$ denotes the absolute value function.

Q. 22: Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Express $\mathbf{E}\left[(X-c)^+\right]$ in terms of $\Phi(\cdot)$ where $(x)^+ = \max\{x, 0\}$.

Q. 23: Answer the following questions related to Beta functions. Assume that $a, b > 0$.

(a) Show that $B(a,b) = 2\int_0^{\pi/2} (\sin\theta)^{2a-1}(\cos\theta)^{2b-1}\mathrm{d}\theta$.

(b) Show that $B(1,1) = 1$ and $B(\frac{1}{2}, \frac{1}{2}) = \pi$.

(c) Show that $B(a,1) = \frac{1}{a}$.

(d) Show the following recursion assuming $b > 1$.

$$B(a,b) = \frac{b-1}{a+b-1}B(a,b-1)$$

Q. 24: The median of a continuous random variable having cumulative distribution function F is that value m such that $F(m) = 0.5$. That is, a random variable is just as likely to be larger than its median as it is to be smaller. Find the median of X if

(a) X is uniformly distributed over (a, b).

(b) X is a normal random variable with parameters (μ, σ^2).

(c) X is an exponential random variable with parameter λ.

Q. 25: Let Y be a non-negative continuous random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

(a) If $\mathbf{E}[Y] < \infty$, then show that the following result holds.

$$\mathbf{E}[Y] = \int_0^\infty \mathbf{P}(Y > y)\mathrm{d}y$$

(b) Furthermore, if $\mathbf{E}[Y^n] < \infty$ for some integer n, then show that the following result holds.

$$\mathbf{E}[Y^n] = \int_0^\infty ny^{n-1}\mathbf{P}(Y > y)\mathrm{d}y$$

Q. 26: Let $\Psi_X(t) = \log M_X(t)$ where $M_X(t)$ is the moment generating function (MGF) of a random variable X. Show that $\Psi_X''(0) = \mathrm{Var}(X)$.

Q. 27: Let X be a log-normal random variable with parameters (μ, σ^2) i.e., $X = e^Z$ where $Z \sim \mathcal{N}(\mu, \sigma^2)$.

(a) Show that $M_X^{(k)}(0) = M_Z(k)$, where $M_X^{(k)}(\cdot)$ is the k-th derivative of $M_X(\cdot)$.

(b) Use the above result to show that $\mathbf{E}[X] = e^{\mu + \frac{\sigma^2}{2}}$ and $\mathrm{Var}(X) = e^{2\mu}\{e^{2\sigma^2} - e^{\sigma^2}\}$.

Q. 28: A continuous random variable X is said to have a Laplace distribution with parameter λ if its probability density function (PDF) is given as follows.

$$f_X(x) = \frac{\lambda}{2}\exp\left(-\lambda|x|\right),\ \forall x \in \mathbb{R}$$

(a) Show that $|X| \sim \exp(\lambda)$.

Q. 29: Let X be a Rayleigh distributed random variable whose PDF is defined below, where $\sigma^2 > 0$.

$$f_X(x) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right),\ \forall x \in \mathbb{R}$$

(a) Prove that the m-th raw moment of X is given by the following expression.

$$\mathbf{E}[X^m] = 2^{m/2}\sigma^m\Gamma\left(1 + \frac{m}{2}\right)$$

(b) Show that the moment generating function (MGF) of X is given by the following expression.

$$M_X(t) = 1 + \sigma t\sqrt{2\pi}\exp\left(\frac{\sigma^2 t^2}{2}\right)\Phi(\sigma t)$$