Probability and Stochastic Processes: Problem Set 4

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Q. 1: Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{G})$ be two measurable spaces. A function $X : \Omega \to S$ is called $\mathcal{F}/\mathcal{G}$-measurable if $X^{-1}(B) \in \mathcal{F}$, $\forall B \in \mathcal{G}$. Let $\mathcal{P} \subset \mathcal{G}$ be such that $\sigma(\mathcal{P}) = \mathcal{G}$.

(a) Prove that $X : \Omega \to S$ is $\mathcal{F}/\mathcal{G}$-measurable if and only if $X^{-1}(B) \in \mathcal{F}$, $\forall B \in \mathcal{P}$. You may use an argument similar to that used in **Q. 1** of P. Set 3.

(b) Prove that $X : \Omega \to \mathbb{R}^n$ is $\mathcal{F}/\mathcal{B}(\mathbb{R}^n)$-measurable if and only if the following holds $\forall a_1, \cdots, a_n \in \mathbb{R}$.
$$X^{-1}\left((a_1, \infty) \times \cdots \times (a_n, \infty)\right) \in \mathcal{F}$$

(c) Let $X_i : \Omega \to \mathbb{R}$ denote an arbitrary real-valued function, $\forall i \in \{1, \cdots, n\}$. Define $X : \Omega \to \mathbb{R}^n$ such that $X(\omega) = (X_1(\omega), \cdots, X_n(\omega))$, $\forall \omega \in \Omega$. Show that the following holds $\forall B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R})$.
$$X^{-1}(B_1 \times \cdots \times B_n) = X_1^{-1}(B_1) \cap \cdots \cap X_n^{-1}(B_n)$$
Use the above result to prove that X is $\mathcal{F}/\mathcal{B}(\mathbb{R}^n)$-measurable if and only if X_i is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable $\forall i \in \{1, \cdots, n\}$.

Q. 2: Let $(\Omega, \mathcal{F})$ be a measurable space, and $X, Y : \Omega \to \mathbb{R}$ be two arbitrary $\mathcal{F}$-measurable functions. Define $X + Y : \Omega \to \mathbb{R}$ such that $(X + Y)(\omega) = X(\omega) + Y(\omega)$, $\forall \omega \in \Omega$.

(a) The density of $\mathbb{Q}$ in $\mathbb{R}$ states that there exists a rational number between any two distinct real numbers. Use this property to show that the following set equality holds $\forall a \in \mathbb{R}$.
$$\{X + Y > a\} = \bigcup_{p \in \mathbb{Q}} \left\{ \{X > p\} \cap \{Y > a - p\} \right\}$$
Conclude that $X + Y$ is $\mathcal{F}$-measurable.

(b) Define $\min\{X, Y\} : \Omega \to \mathbb{R}$ such that $\min\{X, Y\}(\omega) = \min\{X(\omega), Y(\omega)\}$, $\forall \omega \in \Omega$. Prove that $\min\{X, Y\}$ is $\mathcal{F}$-measurable.

(c) Let X, Y be positive-valued functions, i.e., $X(\omega), Y(\omega) > 0$, $\forall \omega \in \Omega$. Define $X/Y : \Omega \to \mathbb{R}$ such that $(X/Y)(\omega) = X(\omega)/Y(\omega)$, $\forall \omega \in \Omega$. Prove that X/Y is $\mathcal{F}$-measurable.

(d) Let X, Y be positive-valued functions, i.e., $X(\omega), Y(\omega) > 0$, $\forall \omega \in \Omega$. Define $XY : \Omega \to \mathbb{R}$ such that $(XY)(\omega) = X(\omega)Y(\omega)$, $\forall \omega \in \Omega$. Prove that XY is $\mathcal{F}$-measurable.

Q. 3: The joint probability mass function (PMF) of two discrete random variables X and Y, defined on the common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, is given as follows.
$$p_{X,Y}(x, y) = \begin{cases} 0.3 & \text{if } (x, y) = (1, 1) \\ 0.5 & \text{if } (x, y) = (1, 2) \\ 0.1 & \text{if } (x, y) = (2, 1) \\ 0.1 & \text{if } (x, y) = (2, 2) \\ 0 & \text{elsewhere} \end{cases}$$

(a) Compute the marginal PMF of X.

(b) Compute the marginal PMF of Y.

(c) Are X and Y independent?

(d) Compute $\mathbf{E}[X]$, $\mathbf{E}[Y]$, and $\mathbf{E}[XY]$.

(e) Compute the conditional PMF of X given $Y = y$ where $y \in \{1, 2\}$.

(f) Compute the conditional PMF of Y given $X = x$ where $x \in \{1, 2\}$.

(g) Compute the PMF of $Z = X + 2Y$.

(h) Compute $\mathbf{P}(XY \leq 3)$.

Q. 4: Let the joint PDF of (X,Y) is $f_{X,Y}$. Show that the PDF of $Z = X + 2Y$ is given as follows.

$$f_{X+2Y}(t) = \int_{-\infty}^{\infty} f_{X,Y}(t-2y, y)\mathrm{d}y$$

Q. 5: Let (X,Y) be a jointly continuous random vector defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ whose joint PDF is given as follows.

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{if } 0 < x < y,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Show that $f_{X,Y}$ is indeed a valid joint PDF.
(b) Compute the marginal PDF of X.
(c) Compute the marginal PDF of Y.
(d) Are X and Y independent?
(e) Compute $\mathbf{E}[X]$, $\mathbf{E}[Y]$, and $\mathbf{E}[XY]$.
(f) Compute the conditional probability $\mathbf{P}\left(Y > 0.5 | X < 0.5\right)$.
(g) Compute $\mathbf{P}\left(Y > 2X\right)$.
(h) Compute $\mathbf{P}\left(X + Y < 1\right)$.
(i) Compute the conditional probability $\mathbf{P}\left(Y > 2X | X + Y < 1\right)$.
(j) Compute the conditional PDF of Y given $X = x$ where $x \in (0,1)$.
(k) Compute the conditional PDF of X given $Y = y$ where $y \in (0,1)$.
(l) Find the PDF of $Z = X + Y$.
(m) Find the PDF of $Z = X + 2Y$.

Q. 6: The joint PDF of X and Y is given as follows.

$$f_{X,Y}(x,y) = \begin{cases} c(x^2 - y^2)e^{-x} & \text{if } 0 \le x < \infty, -x \le y \le x \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the value of c.
(b) Compute the conditional PDF of Y given $X = x$ where $x \in [0, \infty)$.
(c) Compute the conditional PDF of X given $Y = y$ where $y \in (-\infty, \infty)$.

Q. 7: Let X_1 and X_2 be independent exponential random variables with parameters λ_1 and λ_2 respectively, defined on the common probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

(a) Find the PDF of $Z = X_1/X_2$.
(b) Compute $\mathbf{P}(X_1 < X_2)$.

Q. 8: Let X and Y be two random variables defined on the common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Show the following for two real numbers (s,t).

$$\mathbf{P}\left(X \le s, Y \le t\right) = \mathbf{P}\left(X \le s\right) + \mathbf{P}\left(Y \le t\right) + \mathbf{P}(X > s, Y > t) - 1$$

Q. 9: An ambulance travels back and forth at a constant speed along a road of length L. At a certain moment, an accident occurs at a point uniformly distributed on the road. [That is, the distance of the point from one of the fixed ends of the road is uniformly distributed over $(0, L)$.] Assuming that the ambulance's location at the moment of the accident is also uniformly distributed, and assuming independence of the variables, compute the density function of the distance of the ambulance from the accident.

Q. 10: Let W be a gamma random variable with parameters (t, β), and suppose that conditional on $W = w$, $X_1, X_2, \cdots, X_n$ are independent exponential random variables with rate w. Show that the conditional distribution of W given that $X_1 = x_1, X_2 = x_2, ..., X_n = x_n$ is gamma with parameters $(t + n, \beta + \sum_{i=1}^{n} x_i)$.

Q. 11: Let X_1, X_2, X_3 be independent Poisson random variables, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with parameters λ_1, λ_2, and λ_3 respectively. Show that the joint probability mass function of $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$ is given as follows.

$$\mathbf{P}\left[Y_1 = m, Y_2 = n\right] = e^{-(\lambda_1+\lambda_2+\lambda_3)} \frac{\lambda_1^m}{m!} \frac{\lambda_3^n}{n!} \sum_{k=0}^{\min\{m,n\}} k! \binom{m}{k} \binom{n}{k} \left(\frac{\lambda_2}{\lambda_1 \lambda_3}\right)^k$$

where (m, n) are non-negative integers. The above PMF is known as the bivariate Poisson distribution.

Q. 12: Let X and Y be independent random variables, both uniformly distributed over $(0, 1)$. Consider the random variables $R = \sqrt{X^2 + Y^2}$ and $\Theta = \tan^{-1}(Y/X)$.

(a) What are the support sets of the density functions of R and Θ (i.e., the range of values of r, θ for which $f_R(r), f_\Theta(\theta) > 0$)?

(b) Calculate the joint PDF of R and Θ.

(c) Prove that the marginal PDF of R is given as follows, and verify that it is indeed a valid PDF.

$$f_R(r) = \begin{cases} \frac{\pi r}{2} & \text{if } 0 \le r < 1 \\ 2r \sin^{-1}\left(\frac{1}{r}\right) - \frac{\pi r}{2} & \text{if } 1 \le r \le \sqrt{2} \\ 0 & \text{elsewhere} \end{cases}$$

(d) Show that the marginal PDF of Θ is given as follows, and verify that it is indeed a valid PDF.

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2}\sec^2\theta & \text{if } \theta \in [0, \frac{\pi}{4}) \\ \frac{1}{2}\text{cosec}^2\theta & \text{if } \theta \in [\frac{\pi}{4}, \frac{\pi}{2}) \\ 0 & \text{elsewhere} \end{cases}$$

Q. 13: If X_1 and X_2 are independent exponential random variables, each having parameter λ, then show that the joint PDF of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$ is given as follows.

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{\lambda^2}{y_2} e^{-\lambda y_1}, & \text{if } y_1 \ge \ln(y_2), y_2 \ge 1 \\ 0 & \text{elsewhere} \end{cases}$$

Verify that the above function is indeed a valid PDF.

Q. 14: Define a random variable $T = Z/\sqrt{Y/n}$, where $Z \sim \mathcal{N}(0, 1)$, and $Y \sim \chi_n^2$ are independent.

(a) Obtain the conditional PDF of T given $Y = y$, where $y > 0$.

(b) Obtain the joint PDF of (T, Y).

(c) Show that the PDF of T is given by the following expression. This is known as the t-distribution with n-degrees of freedom. This has important applications in statistical inference.

$$f_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}, \quad -\infty < t < \infty$$

Q. 15: Answer the following questions.

(a) Let X and Y be two random variables such that $Y = a + bX$. Show that

$$\rho(X, Y) = \begin{cases} +1 & \text{if } b > 0 \\ -1 & \text{if } b < 0 \end{cases}$$

(b) If $Z \sim \mathcal{N}(0, 1)$, then show that $\text{Cov}(Z^m, Z^n) = 0$ where m is odd and n is even integer.

(c) Let X and Y be independent random variables with means μ_1, μ_2 and variances σ_1^2, σ_2^2 respectively. Show that $\mathbf{E}[(X - Y)^2] = (\mu_1 - \mu_2)^2 + \sigma_1^2 + \sigma_2^2$.

(d) Let $Z \sim \mathcal{N}(0, 1)$ and $Y = a + bZ + cZ^2$. Show that $\rho(Y, Z) = b/\sqrt{b^2 + 2c^2}$.

(e) Show that the following holds for any two random variables X and Z.

$$\mathbf{E}\left[(X - \mathbf{E}[X|Z])^2\right] = \mathbf{E}\left[X^2\right] - \mathbf{E}\left[(\mathbf{E}[X|Z])^2\right]$$

Q. 16: Let X and Y be Bernoulli random variables defined on the common probability space $(\Omega, \mathcal{F}, \mathbf{P})$, i.e., they take values in $\{0, 1\}$. Prove that if X and Y are uncorrelated, then they must be independent. This is one of the few examples where zero correlation implies independence. This implication is not true in general.

Q. 17: Answer the following questions.

(a) Show that $\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, \mathbf{E}[Y|X])$.

(b) Show that if $\mathbf{E}[Y|X]$ is a constant function of X, then X, Y have zero covariance.

(c) Show that if X, Y are identically distributed but not necessarily independent, then $\mathrm{Cov}(X+Y, X-Y) = 0$.

(d) Let $\{(X_j, Y_j)\}_{j=1}^{l}$ be independent and identically distributed random vectors. Note that X_j, Y_j can be dependent. However, X_j and Y_k are independent when $j \neq k$. Define the following parameters.

$$\mathbf{E}[X_j] = \mu_X,\ \mathbf{E}[Y_j] = \mu_Y,\ \mathrm{Var}[X_j] = \sigma_X^2,\ \mathrm{Var}[Y_j] = \sigma_Y^2,\ \rho(X_j, Y_j) = \rho_{X,Y}$$

Compute $\rho\left(\sum_{j=1}^{l} X_j, \sum_{j=1}^{l} Y_j\right)$ in terms of the above parameters.

Q. 18: The number of people who enter an elevator on the ground floor is a Poisson random variable with mean λ. Assume that there are N floors above the ground floor and that each person is equally likely to get off at any one of the N floors, independently of where the others get off. Let X denote the number of people who enter the elevator on the ground floor, and S denote the number of stops the elevator makes before discharging all of its passengers. Our goal is to obtain $\mathbf{E}[S]$. Let $\forall j \in \{1, \cdots, N\}$, the random variable I_j be defined as:

$$I_j = \begin{cases} 1 & \text{if at least one person gets off at the } j\text{th floor} \\ 0 & \text{if no one gets off at the } j\text{th floor} \end{cases}$$

(a) How are S and $\{I_j\}_{j=1}^{N}$ related?

(b) Compute $\mathbf{E}[I_j|X = k]$.

(c) Using the linearity of the expectation operation, compute $\mathbf{E}[S|X = k]$.

(d) Show that $\mathbf{E}[S] = N(1 - e^{-\lambda/N})$. What is the limiting value of $\mathbf{E}[S]$ as $N \to \infty$?

Q. 19: Assuming that X, Y are jointly continuous random variables, answer the following questions.

(a) Find the value of a that minimizes $\mathrm{E}[(a - X)^2]$.

(b) Find the values of a and b that minimize $\mathbf{E}[(a + bX - Y)^2]$.

(c) Find the values of a, b, and c that minimize $\mathbf{E}[(a + bX + cX^2 - Y)^2]$.

Q. 20: Let $X \sim \mathcal{N}(0, 1)$ and $I \sim \mathrm{Bernoulli}(\frac{1}{2})$. Assume that X, I are independent. Define:

$$Y = \begin{cases} X & \text{if } I = 1 \\ -X & \text{if } I = 0 \end{cases}$$

(a) Show that $Y \sim \mathcal{N}(0, 1)$. Are Y and I independent?

(b) Show that $\mathrm{Cov}(X, Y) = 0$.

Q. 21: Let X, Z be independent, $X \sim \mathcal{N}(\mu, 1)$, and $Z \sim \mathcal{N}(0, 1)$.

(a) Show that $\mathbf{E}[\Phi(X)] = \Pr(Z < X)$ where $\Phi(\cdot)$ denotes the CDF of Z.

(b) What is the PDF of $X - Z$?

(c) Show that $\mathbf{E}[\Phi(X)] = \Phi(\mu/\sqrt{2})$.

Q. 22: Suppose that Y is uniformly distributed over $(0, 1)$, and that the conditional distribution of X given $Y = y$ is uniform over $(0, y)$, where $y \in (0, 1)$.

(a) Find the PDF and CDF of X.

(b) Find $\mathbf{E}[X]$ and $\mathrm{Var}(X)$.

(c) Obtain $\mathrm{Cov}(X, Y)$.

Q. 23: The conditional covariance of X, Y, given Z is defined as follows.

$$\mathrm{Cov}(X, Y|Z) = \mathbf{E}\left[(X - \mathbf{E}[X|Z])(Y - \mathbf{E}[Y|Z])|Z\right]$$

(a) Show that the conditional covariance can be expressed as follows.

$$\mathrm{Cov}(X, Y|Z) = \mathbf{E}[XY|Z] - \mathbf{E}[X|Z]\mathbf{E}[Y|Z]$$

(b) Establish the following equality.

$$\mathrm{Cov}(X, Y) = \mathbf{E}\left[\mathrm{Cov}(X, Y|Z)\right] + \mathrm{Cov}\left(\mathbf{E}[X|Z], \mathbf{E}[Y|Z]\right)$$

(c) Substitute $X = Y$ in the above relation, and prove the following.

$$\mathrm{Var}(X) = \mathbf{E}\left[\mathrm{Var}(X|Z)\right] + \mathrm{Var}\left(\mathbf{E}[X|Z]\right)$$

Q. 24: Let $(\Omega, 2^\Omega, \mathbf{P})$ be a probability space where $\Omega = \{a, b, c\}$, and $\mathbf{P} : 2^\Omega \to [0, 1]$ is such that $\mathbf{P}(\{a\}) = \mathbf{P}(\{b\}) = 0.25$, and $\mathbf{P}(\{c\}) = 0.5$. Define two random variables X, Y on $(\Omega, 2^\Omega, \mathbf{P})$ such that:

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = a \\ 2 & \text{if } \omega \in \{b, c\} \end{cases} \qquad Y(\omega) = \begin{cases} 1 & \text{if } \omega = a \\ 2 & \text{if } \omega = b \\ 3 & \text{if } \omega = c \end{cases}$$

(a) Write the joint PMF of X and Y. Are X, Y independent?

(b) Compute $\rho(X, Y)$.

(c) Express $\mathbf{E}[X|Y]$ as a function from Ω to $\mathbb{R}$. In other words, compute $\mathbf{E}[X|Y](\omega)$ for $\omega \in \Omega$.

(d) Show that $\mathbf{E}[\mathbf{E}[X|Y]] = \mathbf{E}[X]$ using the above result.

(e) Show that $\mathbf{E}[X|Y]$ and $\mathbf{E}[X|Y^3]$ denote the same random variable.

(f) Obtain $\sigma(Y)$ (the σ-algebra generated by Y) and show that it is identical to $\sigma(Y^3)$.

(g) Consider the event $A = \{a, c\}$. Compute the conditional expectation $\mathbf{E}[X|A]$.

(h) Express $\mathrm{Var}(X|Y)$ as a function from Ω to $\mathbb{R}$.

Q. 25: Let X and Y be two random variables defined on the common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The moment generating function (MGF) of X is given by $M_X(t) = \exp(2e^t - 2)$ and that of Y by $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$, $\forall t \in \mathbb{R}$. Compute the following quantities if X, Y are independent.

(a) $\mathbf{E}[XY]$

(b) $\mathbf{P}(XY = 0)$.

(c) $\mathbf{P}(X + Y = 1)$.