Probability and Stochastic Processes: Problem Set 5

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Q. 1: Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that almost surely converges to another random variable X, defined on the same probability space. Define the following.

$$A = \left\{\omega \in \Omega \,\Big|\, \lim_{n\to\infty} X_n(\omega) \neq X(\omega)\right\}$$

Let $\epsilon > 0$ be arbitrary. Define the following events $\forall n, k \in \mathbb{N}$.

$$B_n = \left\{\omega \in \Omega \,\Big|\, |X_n(\omega) - X(\omega)| > \epsilon\right\} \text{ and } E_k = \cup_{n\geq k} B_n$$

(a) Argue that $\mathbf{P}(A) = 0$.
(b) Prove that $\cap_{k\in\mathbb{N}} E_k \subset A$.
(c) Use (a) and (b) to conclude that $\mathbf{P}(\cap_{k\in\mathbb{N}} E_k) = 0$.
(d) Argue that $\{E_k\}_{k\in\mathbb{N}}$ is a decreasing sequence of events.
(e) Use the continuity of the probability measure to establish that $\lim_{k\to\infty} \mathbf{P}(E_k) = 0$.
(f) Argue that $B_k \subset E_k$.
(g) Use (e) and (f) to conclude that $\lim_{k\to\infty} \mathbf{P}(B_k) = 0$.
(h) Conclude that $X_n \xrightarrow{P} X$.

The above exercise establishes that almost sure convergence implies convergence in probability.

Q. 2: The goal of this exercise is to show that convergence in probability implies convergence in distribution. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, that converges in probability to another random variable X, also defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let F_{X_n}, F_X denote the CDFs of X_n and X respectively. Assume that F_X is continuous at x. Therefore, for any $\epsilon > 0$, one can always find $\delta > 0$ such that $F_X(x+\delta) - F_X(x) < \epsilon/2$ and $F_X(x) - F_X(x-\delta) < \epsilon/2$.

(a) Prove the following inequality.

$$F_{X_n}(x) \leq F_X(x+\delta) + \mathbf{P}\left(|X_n - X| > \delta\right)$$

(b) Invoke the continuity of F_X at x, and obtain the following inequality.

$$F_{X_n}(x) \leq F_X(x) + \frac{\epsilon}{2} + \mathbf{P}\left(|X_n - X| > \delta\right)$$

(c) Argue that there exists $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$ the following holds.

$$F_{X_n}(x) \leq F_X(x) + \epsilon$$

(d) Prove, in a similar fashion, that there exists $N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$ the following holds.

$$F_{X_n}(x) \geq F_X(x) - \epsilon$$

(e) Combining, show that $\forall n \geq \max\{N_1, N_2\}$, the following holds.

$$|F_{X_n}(x) - F_X(x)| \leq \epsilon$$

(f) Conclude that $X_n \xrightarrow{D} X$.

Q. 3: The goal of this exercise is to show that convergence in expectation implies convergence in probability. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that convergences in expectation to another random variables X, defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Fix an $\epsilon > 0$.

(a) Use Markov's inequality to show that $\mathbf{P}(|X_n - X| > \epsilon) \leq \mathbf{E}[|X_n - X|]/\epsilon$.
(b) Conclude that $(X_n)_{n\in\mathbb{N}}$ converges in probability to X.

Q. 4: Let $(\Omega, 2^\Omega, \mathbf{P})$, be a probability space where $\Omega = \{a, b\}$, and $\mathbf{P} : 2^\Omega \to [0,1]$ is such that $\mathbf{P}(\{a\}) = \mathbf{P}(\{b\}) = 0.5$. Define a random variable $X : \Omega \to \mathbb{R}$ such that $X(a) = 1, X(b) = -1$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that $X_n = (-1)^n X$.

(a) Compute the CDF of X and X_n.

(b) Prove that $X_n \xrightarrow{D} X$.

(c) Let $\epsilon \in (0,1)$. Show the following result.

$$\mathbf{P}\left(|X_n - X| > \epsilon\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

(d) Conclude that $(X_n)_{n\in\mathbb{N}}$ does not converge to X in probability. Therefore, in general, convergence in distribution does not imply convergence in probability.

Q. 5: Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $\mathbf{P} : \mathcal{F} \to [0,1]$ be such that

$$\mathbf{P}([a,b]) = \mathbf{P}((a,b]) = \mathbf{P}([a,b)) = \mathbf{P}((a,b)) = b - a,\ 0 \le a < b \le 1$$

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that

$$X_n(\omega) = \begin{cases} \sqrt{n} & \text{if } \omega \in \left[0, \frac{1}{n}\right] \\ 0 & \text{if } \omega \in \left(\frac{1}{n}, 1\right] \end{cases}$$

Moreover, let X be a zero random variable, i.e., $X(\omega) = 0$, $\forall \omega \in [0,1]$.

(a) Show the following result.

$$\left\{\omega \in \Omega \middle| \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} = (0,1]$$

(b) Conclude that $X_n \xrightarrow{\text{a.s.}} X$.

(c) Conclude that $X_n \xrightarrow{P} X$.

(d) Show that $\mathbf{E}[|X_n - X|^2] = 1$, $\forall n \in \mathbb{N}$.

(e) Conclude that $(X_n)_{n\in\mathbb{N}}$ does not converge to X in expectation. Therefore, in general, almost sure convergence does not imply convergence in expectation. Also, in general, convergence in probability does not imply convergence in expectation.

Q. 6: Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $\mathbf{P} : \mathcal{F} \to [0,1]$ be such that

$$\mathbf{P}([a,b]) = \mathbf{P}((a,b]) = \mathbf{P}([a,b)) = \mathbf{P}((a,b)) = b - a,\ \ 0 \le a \le b \le 1$$

Consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that $X_n(\omega) = \omega^n$, $\forall \omega \in [0,1]$. Let X be another random variable, defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$, such that $X(\omega) = 0$, $\forall \omega \in [0,1]$.

(a) Show that the sequence $(X_n)_{n\in\mathbb{N}}$ converges almost surely to X.

(b) Show that $(X_n)_{n\in\mathbb{N}}$ does not converge pointwise to X. Therefore, in general, almost sure convergence does not imply pointwise convergence.

Q. 7: Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and $\mathbf{P} : \mathcal{F} \to [0,1]$ be such that $\mathbf{P}([a,b]) = b - a$, $0 \le a \le b \le 1$. Define a sequence of intervals $(I_n)_{n\in\mathbb{N}}$ such that

$$I_n = \left[\frac{r}{2^p}, \frac{r+1}{2^p}\right],\ \text{where } p = \lfloor \log_2 n \rfloor,\ \text{and } r = n - 2^p$$

For example, the first few members of the sequence are given as follows.

$$I_1 = [0,1]\,, I_2 = \left[0, \frac{1}{2}\right],\ I_3 = \left[\frac{1}{2}, 1\right],$$
$$I_4 = \left[0, \frac{1}{4}\right],\ I_5 = \left[\frac{1}{4}, \frac{1}{2}\right], I_6 = \left[\frac{1}{2}, \frac{3}{4}\right], I_7 = \left[\frac{3}{4}, 1\right], \text{etc.}$$

The sequence $(I_n)_{n\in\mathbb{N}}$ is called a *typewriter* sequence. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $X_n = \mathbf{1}(I_n)$, $\forall n \in \mathbb{N}$, where $\mathbf{1}$ is an indicator function, and X be another random variable on $(\Omega, \mathcal{F}, \mathbf{P})$ such that $X(\omega) = 0$, $\forall \omega \in [0,1]$.

(a) Show that $(X_n)_{n\in\mathbb{N}}$ converges to X in expectation.

(b) Show that $(X_n)_{n\in\mathbb{N}}$ does not almost surely converge to X. Therefore, convergence in expectation does not imply almost sure convergence.

Q. 8: Let X_0 be a random variable defined on the probability space $(\Omega, \mathbf{F}, \mathbf{P})$ such that $\mathbf{P}(X_0 \geq 0) = 1$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that

$$X_n = \sqrt{X_{n-1}} + 6,\ \forall n \in \mathbb{N}$$

(a) Argue that $\mathbf{P}(X_n \geq 0) = 1, \forall n \in \mathbb{N}$. Therefore, the above recursion is well-defined almost surely.

(b) Prove that for any $x \geq 0$, the following holds: $|6 + \sqrt{x} - 9| \leq \frac{1}{3}|x - 9|$.

(c) Argue that $|X_n(\omega) - 9| \leq \frac{1}{3}|X_{n-1}(\omega) - 9|, \forall \omega \in \Omega, \forall n \in \mathbb{N}$.

(d) Argue that $|X_n(\omega) - 9| \leq 3^{-n}|X_0(\omega) - 9|, \forall \omega \in \Omega, \forall n \in \mathbb{N}$.

(e) Prove that $X_n \xrightarrow{\text{a.s.}} 9$.

(f) Prove that $X_n \xrightarrow{E} 9$.

Q. 9: Let X be a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $a > 0$.

(a) If $X \sim \text{Gamma}(\alpha, \lambda)$, $\alpha, \lambda > 0$, prove via Chernoff's bound that the following holds $\forall a > \alpha/\lambda$.

$$\mathbf{P}(X \geq a) \leq \left(\frac{\lambda a}{\alpha}\right)^{\alpha} \exp\left(\alpha - a\lambda\right)$$

(b) Using the above result, find an upper bound of $\mathbf{P}(X \geq a)$, where $X \sim \exp(\lambda)$, $\lambda > 0$, and $a > \lambda^{-1}$.

(c) Using the above result, find an upper bound of $\mathbf{P}(X \geq a)$, where $X \sim \chi^2_n$ (chi-squared distribution with n degrees of freedom), and $a > n$.

(d) If $X \sim \text{Binomial}(n, p)$, and $np < a < n$, then prove the following using Chernoff's bound.

$$\mathbf{P}(X \geq a) \leq \frac{n^n p^a (1-p)^{n-a}}{a^a (n-a)^{n-a}}$$

Q. 10: Write a Python code to perform the following operations.

(a) Generate $N = 10^4$ IID samples of a uniform random variable in $(0, 1)$.

(b) Empirically calculate the PDF of the obtained samples. You may divide the range $(0, 1)$ into $M = 10^2$ bins, and compute the value of the PDF in each bin. Plot the obtained PDF. Verify whether it indeed looks like the PDF of a uniform random variable in $(0, 1)$.

(c) Apply the transformation $f(u) = -\ln(1 - u)$ to each sample, and then empirically obtain their PDF in the range $(0, 10)$. You may divide this range into $K = 10^2$ bins, and compute PDF values for each bin. Plot the obtained PDF.

(d) Theoretically, what should it look like? Does it match your expectations?

Q. 11: A fair die is thrown 1000 times. If X_n denotes the outcome of the nth throw, then find the approximate probability of the event that $X_1 X_2 \cdots X_{1000} > 3^{1000}$. You may assume that X_n's are independent.

Q. 12: Simulating a continuous random variable via the usual technique of inverting a CDF may not always be efficient. For Gaussian random variables, however, we can use an alternative process known as the Box-Muller method. Let $U_1, U_2 \sim \text{Unif}(0, 1)$ be independent. Define random variables X_1 and X_2 as follows.

$$X_1 = \sqrt{-2\log U_1}\cos(2\pi U_2),$$
$$X_2 = \sqrt{-2\log U_1}\sin(2\pi U_2)$$

Prove that X_1 and X_2 are independent standard Gaussian random variables.