Probability and Stochastic Processes: Problem Set 6

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Q. 1: Let $\{X_t\}_{t\in\mathbb{T}}$ be a (real-valued) wide sense stationary (WSS) process, where either $\mathbb{T}=\mathbb{Z}$ or $\mathbb{T}=\mathbb{R}$.

(a) Show that the autocorrelation function R_X is an even function, i.e., $R_X(\tau)=R_X(-\tau)$, $\forall\tau\in\mathbb{T}$.

(b) Show that $R_X(0)\geq|R_X(\tau)|$, $\forall\tau\in\mathbb{T}$. To prove this result, you may use the following inequality.

$$\mathbf{E}\left[\left\{aX(t)-X(t+\tau)\right\}^2\right]\geq 0,\ \forall a\in\mathbb{R}$$

(c) Suppose that $\{X_t\}_{t\in\mathbb{T}}$ is periodic with period T, i.e., $X_t=X_{t+T}$ almost surely, $\forall t\in\mathbb{T}$. Prove that the autocorrelation function is also periodic with period T, i.e., $R_X(\tau)=R_X(\tau+T)$, $\forall\tau\in\mathbb{T}$. You may apply the result that if X and Y are random variables defined on the same probability space such that $X=Y$ almost surely, then $\mathbf{E}[X]=\mathbf{E}[Y]$.

Q. 2: Let $X=\{X_t\}_{t\in\mathbb{R}}$ be a random process such that $X_t=R-St$, $\forall t\in\mathbb{R}$ where R and S are independent Rayleigh distributed random variables with parameters σ_R^2 and σ_S^2, respectively. Mathematically, the PDFs of R and S are expressed as follows.

$$f_R(r)=\begin{cases}\dfrac{r}{\sigma_R^2}\exp\left(-\dfrac{r^2}{2\sigma_R^2}\right) & \text{if } r\geq 0\\ 0 & \text{if } r<0\end{cases},\ \text{ and }\ f_S(s)=\begin{cases}\dfrac{s}{\sigma_S^2}\exp\left(-\dfrac{s^2}{2\sigma_S^2}\right) & \text{if } s\geq 0\\ 0 & \text{if } s<0\end{cases}$$

(a) Indicate three typical sample paths of X in a single sketch. Describe in words the set of possible sample paths of X.

(b) Show that X is not a wide sense stationary (WSS) process. Investigate whether X is Markovian.

(c) Prove that X has stationary increments, i.e., $\forall s,t\in\mathbb{R}$, the distribution of $X_{t+s}-X_s$ is independent of s. Additionally show that X does not have independent increments, i.e., if $s_1<t_1<\cdots<s_n<t_n$, then the random variables $X_{t_1}-X_{s_1},\cdots,X_{t_n}-X_{s_n}$ are not independent.

(d) Let A denote the area of the triangle bounded by the coordinate axes and the graph of X. Prove that

$$\mathbf{E}[A]=\frac{\sigma_R^2}{\sigma_S}\sqrt{\frac{\pi}{2}}$$

Q. 3: Define a stochastic process $X=\{X_t\}_{t\in\mathbb{R}}$ such that $X_t=A\cos(\Omega t+\Theta)$, $\forall t\in\mathbb{R}$, where A is a random variable with mean 2 and variance 4, $\Omega\in\mathrm{Unif}(0,4\pi)$, and $\Theta\in\mathrm{Unif}(0,2\pi)$. Also, A,Ω,Θ are independent.

(a) Find the mean and autocorrelation functions of X. Use this result to prove that X is a wide sense stationary (WSS) process.

(b) Argue that the tuple (A,Ω,Θ) has the same distribution as (A,Ω,Θ'), where $\Theta'=(\Theta+\Omega h)\bmod 2\pi$ for some constant h. Use this result to prove that X is a stationary process.

Q. 4: Let $X=\{X_t\}_{t\in\mathbb{N}}$ be a discrete time autoregressive process of order 1, often abbreviated as AR(1), which is recursively defined as follows $\forall t\in\mathbb{N}$.

$$X_t=\alpha X_{t-1}+\epsilon_t$$

where $\alpha\in(-1,1)$, $X_0,\epsilon_1,\epsilon_2,\cdots$ are independent random variables, and $\epsilon_t\sim\mathcal{N}(0,\sigma^2)$, $\forall t\in\mathbb{N}$.

(a) Argue that X is a Markov process.

(b) If $X_0=0$ almost surely, then calculate $\mathbf{E}[X_t]$ and $\mathrm{Var}(X_t)$, and prove that X cannot be a wide sense stationary (WSS) process.

(c) If $X_0\sim\mathcal{N}(0,\sigma^2/(1-\alpha^2))$, then calculate $\mu_X(t)$ and $R_X(t,t+s)$, where $t\in\mathbb{N}$ and $s\in\{0,1,\cdots\}$, and conclude that X is a wide sense stationary (WSS) process.

Q. 5: Let $X_1, X_2, \cdots$ be independent and identically distributed (IID) random variables with the common cumulative distribution function (CDF) F_X. Fix an arbitrary $x \in \mathbb{R}$, and define the empirical CDF value of n observations at x as follows.

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i \leq x)$$

(a) Prove the following results $\forall n \in \mathbb{N}$, and $\forall r \in \{0, 1, \cdots\}$, and comment on whether $\{\hat{F}_n(x)\}_{n\in\mathbb{N}}$ is a wide sense stationary (WSS) process.

$$\mathbf{E}[\hat{F}_n(x)] = F_X(x) \text{ and } \mathrm{Cov}[\hat{F}_n(x), \hat{F}_{n+r}(x)] = \frac{F_X(x)(1 - F_X(x))}{n + r}$$

(b) Prove that the sequence $(\hat{F}_n(x))_{n\in\mathbb{N}}$ converges almost surely to $F_X(x)$.

(c) Let $Y_n(x) = \sqrt{n}(\hat{F}_n(x) - F_X(x)), \forall n \in \mathbb{N}$. Establish that the sequence $(Y_n(x))_{n\in\mathbb{N}}$ converges in distribution to $Z \sim \mathcal{N}(0, \sigma^2)$, where $\sigma^2 = F_X(x)(1 - F_X(x))$.

Q. 6: Let $\{U_n\}_{n\in\mathbb{Z}}$ indicate a collection of independent random variables, each uniformly distributed over $(0, 1)$. Define a random process $X = \{X_n\}_{n\in\mathbb{Z}}$, where $X_n = \max\{U_n, U_{n-1}\}, \forall n \in \mathbb{Z}$.

(a) Prove that X is a stationary process.

(b) Prove that the first-order CDF of X is given as follows $\forall t_1 \in \mathbb{Z}$ and $\forall x_1 \in \mathbb{R}$.

$$F_{X,1}(x_1, t_1) = (F_U(x_1))^2 \text{ where } F_U(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

(c) Prove that the second-order CDF of X is given as follows $\forall t_1, t_2 \in \mathbb{Z}$, and $\forall x_1, x_2 \in \mathbb{R}$.

$$F_{X,2}(x_1, t_1; x_2, t_2) = \begin{cases} (F_U(\min\{x_1, x_2\}))^2 & \text{if } t_1 = t_2 \\ F_U(x_1)F_U(\min\{x_1, x_2\})F_U(x_2) & \text{if } |t_1 - t_2| = 1 \\ (F_U(x_1))^2(F_U(x_2))^2 & \text{if } |t_1 - t_2| > 1 \end{cases}$$

(d) Let $x \in (0, 1)$. Show the following results $\forall t \in \mathbb{Z}$, where $\mathbf{P}$ is the underlying probability measure.

$$\mathbf{P}(X_{t+1} > x \mid X_t > x) = 1 - \frac{x^2}{1 + x} \text{ and } \mathbf{P}(X_{t+1} > x \,|X_t > x, X_{t-1} < x) = 1$$

Conclude that X cannot be a Markovian process.

Q. 7: Let $(U_n)_{n=0}^{\infty}$ be a sequence of independent uniformly distributed random variables on $(0, 1)$. Define a continuous-time random process $X = \{X_t\}_{t\geq 0}$ such that

$$X_t = (1 - \alpha)U_n + \alpha U_{n+1}, \text{ where } n = \lfloor t \rfloor, \alpha = t - \lfloor t \rfloor$$

(a) Calculate $\mathbf{E}[X_t]$ and $\mathrm{Var}(X_t)$, and conclude that X cannot be a wide sense stationary (WSS) process.

(b) Prove that the first-order PDF of X is as follows $\forall x \in \mathbb{R}, \forall t \geq 0$ such that $\alpha = t - \lfloor t \rfloor \in (0, 0.5)$.

$$f_{X,1}(x, t) = \begin{cases} \dfrac{x}{\alpha(1 - \alpha)} & \text{if } 0 < x < \alpha \\ \dfrac{1}{1 - \alpha} & \text{if } \alpha \leq x < 1 - \alpha \\ \dfrac{1 - x}{\alpha(1 - \alpha)} & \text{if } 1 - \alpha \leq x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(c) Find the probability of the event that $\max_{0\leq t\leq 10} X_t \leq 0.5$.

Q. 8: A mouse performs a random walk on the three vertices of a triangle, which are indexed as $0, 1$ and 2. Let the random walk be represented by the discrete time stochastic process $X = \{X_t\}_{t=0}^{\infty}$, where $X_t \in \{0, 1, 2\}$ denotes the index of the vertex occupied by the mouse at time t. At each instant, the mouse jumps to one of the two neighboring vertices, each with probability 0.5.

(a) Show that X is a Markov chain and write down its one-step transition probability matrix, P.

(b) Establish that X is ergodic, and determine its stationary distribution, π. Numerically verify that each row of P^n approaches π as $n \to \infty$.

Q. 9: Let $\{N_t\}_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

(a) Prove that the probability that $N_2 = 2$ given $N_1 \geq 1$ can be expressed as follows.

$$\mathbf{P}(N_2 = 2 \mid N_1 \geq 1) = \frac{3\lambda^2 e^{-2\lambda}}{2(1-e^{-\lambda})}$$

For the following questions, assume that S_n denotes the instance of occurrence of the nth event.

(b) Show that the joint PDF of $(S_1, \cdots, S_n)$ can be expressed as follows.

$$f_{S_1,\cdots,S_n}(s_1, \cdots, s_n) = \begin{cases} \lambda^n \exp(-\lambda s_n) & \text{if } 0 < s_1 < \cdots < s_n \\ 0 & \text{elsewhere} \end{cases}$$

(c) If $0 < s_1 < \cdots < s_n < T$, then show that the probability that $N_T = n$ given $S_1 = s_1, \cdots, S_n = s_n$ can be expressed as follows.

$$\mathbf{P}(N_T = n \mid S_1 = s_1, \cdots, S_n = s_n) = \exp(-\lambda(T - s_n))$$

(d) Prove that the conditional joint PDF of $(S_1, \cdots, S_n)$ given $N_T = n$ can be expressed as follows.

$$f_{S_1,\cdots,S_n|N_T}(s_1, \cdots, s_n|n) = \begin{cases} \dfrac{n!}{T^n} & \text{if } 0 < s_1 < \cdots < s_n < T \\ 0 & \text{elsewhere} \end{cases}$$